Math 115 – Global Extrema Problems

Optimization, i.e. finding the global minimum and/or global maximum values of a function, is a very important application of calculus.

> As you read these examples, take note of the highlighted **required** key features in each solution. Notice that the work required when finding **global** extrema is typically quite different from the work required when finding **local** extrema.

Example 1

Let

$$f(x) = \begin{cases} \dfrac{2-x}{e} & x < 1 \\ x^2 e^{-x} & x \geqslant 1. \end{cases}$$

Find the x-coordinate(s) of all global extrema of $f(x)$ on $[-3, 3]$. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

check continuity

f is continuous, since $\frac{2-1}{e} = \frac{1}{e}$

and $1^2 e^{-1} = \frac{1}{e}$

check differentiability

$$f'(x) = \begin{cases} -\frac{1}{e} & x < 1 \\ x^2(-e^{-x}) + 2xe^{-x} & x > 1 \end{cases}$$

$$= \begin{cases} -\frac{1}{e} & x < 1 \\ xe^{-x}(2-x) & x > 1 \end{cases}$$

<u>$f'(x)$ DNE</u> at $x=1$ since

$1 \cdot e^{-1}(2-1) = \frac{1}{e} \neq -\frac{1}{e}$

find critical points algebraically

<u>$f'(x) = 0$</u>: $-\frac{1}{e}$? never zero

$xe^{-x}(2-x) = 0$ when

$x = 0$ or 2

↳ not in domain of this piece

So $x = 1, 2$ are the critical points

consider values of f at critical points and endpoints of interval

x	$f(x)$
-3	1.839
1	0.368
2	0.541
3	0.448

By EVT, f has a global max at $x = -3$ and a global min at $x = 1$.

> - Don't forget to check for continuity and differentiability, particularly for piecewise functions.
> - Find critical points algebraically. **Don't forget** to check for points where f' does not exist.
> - Consider the value of f at all relevant critical points.
> - Consider the value of f at the endpoints of the interval.

Example 2a

A function $g(x)$ and its derivative $g'(x)$ are given below.

$$g(x) = \frac{(x-1)^2}{x^3} \qquad\qquad g'(x) = -\frac{(x-1)(x-3)}{x^4}$$

Find the x-coordinate(s) of all global extrema of $g(x)$ on $(0, \infty)$. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

$\underline{g'(x)=0}$: $x = 1, 3$

$\underline{g'(x) \text{ DNE}}$ $x=0$, not in domain

consider values of g at critical points

$g(1) = 0$

$g(3) = \frac{4}{27}$

$\lim_{x\to 0^+} g(x) = \infty$

$\lim_{x\to\infty} g(x) = 0$

consider end behavior

so g has a global min at $x=1$ and no global max

- Consider the value of g at all relevant critical points.
- Consider the behavior of g at each end of the interval.

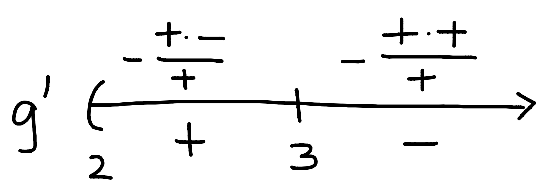
Example 2b

Suppose we had been asked to find the location(s) of all global extrema of $g(x)$ on $(2, \infty)$ instead. We could certainly provide a solution similar to that above. However, because in this case there is **only one** critical point of $g(x)$ in the interval $(2, \infty)$, we can provide the following alternate solution.

$\underline{g'(x)=0}$: $x = 1, 3$, only 3 is in $(2, \infty)$

$\underline{g'(x) \text{ DNE}}$ $x=0$, not in domain

justify whether the one critical point is a local extrema



by 1st Deriv. Test, $x=3$ is a local max.

Since g is continuous on $(2,\infty)$ and there is only one critical point on this interval, the local max at $x=3$ must also be the global max, and there is no global min.

clear conclusion

- Show that the one critical point in the interval is a local max or min, including all necessary justification (see Local Extrema and Inflection Point Problems).
- Clearly state that there is only one critical point in the interval when justifying the conclusion.