Math 115 – Local Extrema and Inflection Point Problems

It is important to be able to classify the critical points of a function as being local maxima or local minima using the 1st Derivative Test and the 2nd Derivative Test. Similarly, it is important to be able to identify a function's inflection points.

> As you read these examples, take note of the highlighted **required** key features in each solution.

Example 1: The 1st Derivative Test using test points

Find the x-coordinate(s) of all local maxima and minima of $f(x) = x^5 - 10x^3 - 8$. You must use calculus to find your answers, and be sure to show enough evidence to fully justify your answers.

$f'(x) = 5x^4 - 30x^2$
$\quad = 5x^2(x^2-6)$

find critical points algebraically

$\underline{f'(x)=0}$: $5x^2=0$, $x=0$
or $x^2-6=0$, $x=\pm\sqrt{6}$

$\underline{f'(x) \text{ DNE}}$: none

1st Deriv. Test:

f': $+$ | $-\sqrt{6}$ | $-$ | 0 | $-$ | $\sqrt{6}$ | $+$

label number line

$f'(-3) = 135$
$f'(-1) = -25$
$f'(1) = -25$
$f'(3) = 135$

give values of f' at test points

so local max at $x=-\sqrt{6}$
local min at $x=\sqrt{6}$

> - Find critical points algebraically.
> - Make sure it's clear what test you're using, and clearly label any number lines or tables.
> - Give values of f' at appropriate test points. Note that it is **not** sufficient to simply indicate, for example, that "$f'(1) < 0$".

Example 1: The 1st Derivative Test using "sign logic"

We now give an alternate solution to the problem above.

$f'(x) = 5x^4 - 30x^2$
$\quad = 5x^2(x^2-6)$

find critical points algebraically

$\underline{f'(x)=0}$: $5x^2=0$, $x=0$
or $x^2-6=0$, $x=\pm\sqrt{6}$

$\underline{f'(x) \text{ DNE}}$: none

1st Deriv. Test:

clearly indicate sign logic

f': $+\cdot+=+$ | $-\sqrt{6}$ | $+\cdot-=-$ | 0 | $+\cdot-=-$ | $\sqrt{6}$ | $+\cdot+=+$

label number line

so local max at $x=-\sqrt{6}$
local min at $x=\sqrt{6}$

> - Find critical points algebraically.
> - Make sure it's clear what test you're using, and clearly label any number lines or tables.
> - For each interval, clearly indicate the signs of all factors or other relevant pieces of f', and the resulting sign of f'.

Example 2: The 2nd Derivative Test

Let $g(x) = x - \ln(x)$. Use the 2nd Derivative Test to classify all critical points of g.

find critical points algebraically

domain $(0, \infty)$

$g'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$

$\underline{g'(x) = 0}$ $x = 1$

$\underline{g'(x) \text{ DNE}}$ $x = 0$, not in domain

$g''(x) = \frac{1}{x^2}$

2^{nd} Deriv. Test:

$g''(1) = 1 > 0$

(or $g''(1) = \frac{+}{+} = +$)

so $x = 1$ is a local min

give value or sign logic of g'' at critical point

- Find critical points algebraically. Don't forget to check for points where g' does not exist.
- Make sure it's clear what test you're using.
- Give values of g'' at the critical points of g, or use appropriate "sign logic." Note that it is **not** sufficient to simply indicate, for example, that "$g''(2) > 0$".

Note: One advantage of the 2nd Derivative Test is that it can be very quick to implement, particularly if g'' is given to you or is easy to find. One disadvantage, however, is that it is not guaranteed to work. For example, if we attempt to use the 2nd Derivative Test on $f(x)$ from Example 1 to classify the critical point $x = 0$, we find that $f''(0) = 0$, so the test is inconclusive; we would need to use the 1st Derivative Test for this critical point instead.

Example 3: Finding Inflection Points

Suppose that $h(x)$ is a continuous function defined for all real numbers whose second derivative is given by $h''(x) = \dfrac{(6x-9)(x-2)^2}{(x+1)^{1/3}}$. Find the x-coordinate(s) of all inflection points of h.

find candidate points algebraically

$\underline{h''(x) = 0}$ $x = \frac{3}{2}, 2$

$\underline{h''(x) \text{ DNE}}$ $x = -1$

$x < -1$: $h''(x) = \frac{- \cdot +}{-} = +$

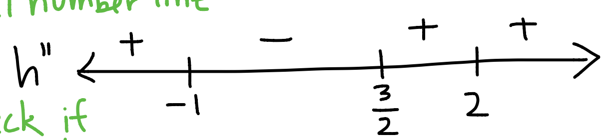
$-1 < x < \frac{3}{2}$: $h''(x) = \frac{- \cdot +}{+} = -$

$\frac{3}{2} < x < 2$: $h''(x) = \frac{+ \cdot +}{+} = +$

$x > 2$: $h''(x) = \frac{+ \cdot +}{+} = +$

Use sign logic, or give values of h'' at test points

label number line



check if concavity changes

(or, $h''(-2) = 336$ $h''(1.75) \approx 0.07$
$h''(1) \approx -2.3$ $h''(3) \approx 5.67$)

so inflection points at $x = -1, \frac{3}{2}$

- Algebraically find the points where h'' is 0 or undefined.
- **Don't forget** to check whether the concavity of h changes at these points: give values of h'' at appropriate test points, or use appropriate "sign logic." Note that it is **not** sufficient to simply indicate, for example, that "$h''(1) < 0$".
- Clearly label any number lines or tables.