Math 105 — First Midterm

October 10, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

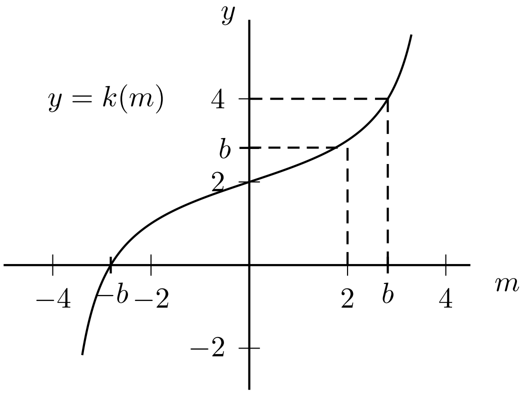
Problem	Points	Score
1	20	
2	10	
3	10	
4	16	
5	16	
6	18	
7	10	
Total	100	

1. [20 points] Use the functions g, h, p, and k given below to answer the questions that follow. *Note: Some answers may involve the constant b.*

t	-4	-2	0	2	4	6
$g(t)$	4	b	2	1	-2	-b

$$h(y) = \frac{2^y}{y^2+1}$$

$$p(x) = \begin{cases} (x+4)^2 - 5 & \text{for } -3 \le x \le -1 \\ 1.2(0.2)^x & \text{for } x > -1 \end{cases}$$



a. [2 points]
Evaluate $p(-1) + p(1)$.

Solution:
$p(-1) + p(1) = \big((-1+4)^2 - 5\big) + 1.2(0.2)^1 = \big((3)^2 - 5)\big) + 1.2(0.2) = 4 + 0.24 = 4.24$

Answer: $\boxed{4.24}$

b. [2 points] Evaluate $p(k(0))$.

Solution: $p(k(0)) = p(2) = 1.2(0.2)^2 = 1.2(0.04) = 0.048$

Answer: $\boxed{0.048}$

c. [2 points] Evaluate $h(g(-2) + 2)$.

Solution: $h(g(-2)+2) = h(b+2) = \dfrac{2^{b+2}}{(b+2)^2+1} = \dfrac{2^{b+2}}{b^2+4b+5}$

Answer: $\boxed{\dfrac{2^{b+2}}{b^2+4b+5}}$

d. [2 points] Solve $k(m) = b$ for m.

Solution: Since $k(2) = b$ (and no other input gives an output of b) we see that the unique solution is $m = 2$.

Answer: $\boxed{m = 2}$

e. [2 points] Assume g and k are invertible. Evaluate $g^{-1}(-2) + k^{-1}(0)$.

Solution: $g^{-1}(-2) + k^{-1}(0) = 4 + (-b) = 4 - b$

Answer: $\boxed{4 - b}$

This problem continues on the next page.

This is a continuation of the problem from the previous page.

Recall that $h(y) = \dfrac{2^y}{y^2+1}$ and $p(x) = \begin{cases} (x+4)^2 - 5 & \text{for } -3 \le x \le -1 \\ 1.2(0.2)^x & \text{for } x > -1. \end{cases}$

f. [3 points] Find the domain of h. Use either inequalities or interval notation to give your answer. Please remember to show your work.

Solution: Since 2^y and $y^2 + 1$ are both defined for all values of y, the only possible restriction is that the denominator cannot be zero. However, $y^2 + 1 > 0$ for all real values of y, so the domain of h is the set of all real numbers.

Domain: $(-\infty, \infty)$

g. [3 points] Find the domain of p. Use either inequalities or interval notation to give your answers. Please remember to show your work.

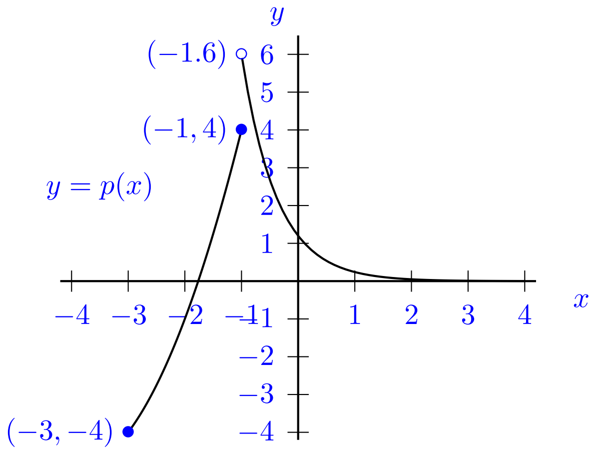
Solution: The first piece of the formula defines the function for values of x in the interval $[-3, -1]$. The second piece does so for values of x in the interval $(-1, \infty)$. Hence the function is defined for all values of x in the interval $[-3, \infty)$

Domain: $[-3, \infty)$ (All real numbers x with $x \ge -3$)

h. [4 points] Find the range of p. Use either inequalities or interval notation to give your answers. Please remember to show your work; this includes sketching any graphs you use.

Solution:

The graph of $y = p(x)$ is shown to the right. From the graph, we see that the smallest output of the function p on its domain is -4. The other outputs are all the real numbers from -4 up to (but not including) 6.



Range: $[-4, 6)$ (All real numbers y with $-4 \le y < 6$)

2. [10 points] The table below gives data about the participation of athletes representing the United States during the Winter Olympic Games since 1994. For each year Y in which the Winter Olympics were held, C is the total number of US competitors, S is the total number of sports in which US athletes competed, E is the total number of different events, and M is the total number of medals won by US competitors at the Olympic games that year.

Y	1994	1998	2002	2006	2010
C	147	186	202	211	216
S	12	14	15	15	15
E	61	68	78	84	86
M	13	13	34	25	37

a. [2 points] In one complete sentence, explain why C is a function of E for the years represented in the table.

Solution: C is a function of E because each value of E (the input) corresponds to one and only one value of C (the output).

b. [3 points] Since C is a function of E, we can write $C = g(E)$. Evaluate the average rate of change of g for E between 68 and 84. *Include units.*

Solution: The average rate of change is $\frac{\Delta C}{\Delta E} = \frac{g(84) - g(68)}{84 - 68} = \frac{211 - 186}{84 - 68} = \frac{25}{16}$ competitors per event.

Answer: $\boxed{\frac{25}{16} \text{ competitors per event}}$

c. [5 points] Based on the data provided in the table, determine which, if any, of the following statements could be true. (There may be none, one, or more than one.)
Circle all of the statements that could be true. *No explanations are required.*

○ C is a decreasing function of E.

○ [✓ C is an increasing function of E.]

○ C is concave up as a function of Y.

○ [✓ C is concave down as a function of Y.]

○ [✓ E is a function of C.]

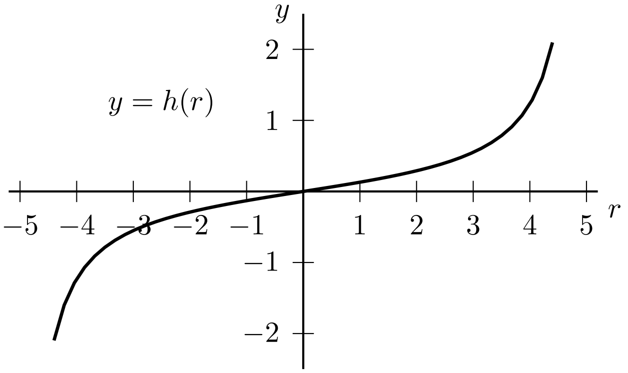
○ E is a function of S.

○ C is a function of M.

○ [✓ Y is a function of C.]

3. [10 points] In each problem below, some information about a function has been given. This is followed by statements about certain characteristics of the function. Choose the ONE best option in each box. You do NOT need to show any work on this problem.

a. [6 points] The graph of a function $h(r)$ is shown below.



Circle ONE	Circle ONE	Circle ONE
○ $\frac{h(4)-h(1)}{3} > \frac{h(4)-h(2)}{2}$ ○ [✓ $\frac{h(4)-h(1)}{3} < \frac{h(4)-h(2)}{2}$] ○ $\frac{h(4)-h(1)}{3} = \frac{h(4)-h(2)}{2}$ ○ The relationship between $\frac{h(4)-h(1)}{3}$ and $\frac{h(4)-h(2)}{2}$ cannot be determined from the information provided.	On the portion of the domain shown, $h(r)$ is ○ [✓ always increasing] ○ always decreasing ○ neither of these	On the portion of the domain shown, $h(r)$ is ○ always concave up ○ always concave down ○ [✓ neither of these]

b. [4 points] The amount of time it takes to cook a giant slab of tofu is a function of the weight of the slab. The more a slab of tofu weighs, the longer it takes for it to fully cook. However, as the weight of the tofu slab goes up, the additional time required per extra pound goes down. Let $B(w)$ be the time, in hours, that it takes to cook a giant slab of tofu weighing w pounds.

Circle ONE	Circle ONE
○ [✓ $B(w)$ is always *increasing.*] ○ $B(w)$ is always *decreasing.* ○ Neither of the above statements is true.	○ $B(w)$ is always *concave up.* ○ [✓ $B(w)$ is always *concave down.*] ○ Neither of the above statements is true.

4. [16 points] Marisa is planning to open a lemonade stand, and she needs to buy equipment and ingredients to make the lemonade. If she decides to make a total of 12 gallons of lemonade, the equipment and ingredients will cost her a total of 57 dollars. However, if she decides to make 20 gallons, it will cost her 85 dollars.

a. [5 points] Let $C(g)$ be the cost to Marisa, in dollars, of producing g gallons of lemonade. Assuming $C(g)$ is a linear function, find a formula for $C(g)$.

Solution: From the information provided in the problem statement, we know $C(12) = 57$ and $C(20) = 85$.
First, we find the average rate of change (slope) of C:

$$\frac{\Delta C}{\Delta g} = \frac{C(20) - C(12)}{20 - 12} = \frac{85 - 57}{20 - 12} = \frac{28}{8} = 3.5$$

Using point-slope form, we then find that $C(g)-57 = 3.5(g-12)$ so $C(g) = 57 + 3.5(g - 12)$.

Answer: $\boxed{C(g) = 57 + 3.5(g - 12) \text{ or (in slope-intercept form) } C(g) = 15 + 3.5g}$

b. [3 points] Find and give a practical interpretation, in the context of this problem, of the slope of the function $C(g)$. *Include units.*

Solution: From part (a) above the slope is 3.5 dollars per gallon.
In the context of this problem, this means that it costs Marisa an additional \$3.50 for the ingredients for each additional gallon of lemonade she decides to make.

c. [2 points] Find the vertical intercept of the function $C(g)$. *Include units.*

Solution: The vertical intercept is $C(0) = 15$ dollars.

Answer: $\boxed{\text{15 dollars}}$

This problem continues on the next page.

This is a continuation of the problem from the previous page. For your convenience, the original problem statement has been reprinted here.

Marisa is planning to open a lemonade stand, and she needs to buy equipment and ingredients to make the lemonade. If she decides to make a total of 12 gallons of lemonade, the equipment and ingredients will cost her a total of 57 dollars. However, if she decides to make 20 gallons, it will cost her 85 dollars.

d. [3 points] Marisa sells lemonade for 25 cents per cup (there are 16 cups in one gallon of lemonade). Assuming she can sell all of the lemonade she makes, find a formula for $R(g)$, the total amount of money (in dollars) Marisa takes in from lemonade sales, i.e. her revenue, if she makes g gallons of lemonade.

Solution: If Marisa sells g gallons of lemonade, then she sells $16g$ cups of lemonade. At 25 cents (or 0.25 dollars) per cup, this means that her revenue (in dollars) is $R(g) = 16g(0.25) = 4g$.

Answer: $\boxed{R(g) = 4g}$

e. [3 points] What is the minimum number of gallons of lemonade Marisa needs to make in order not to lose money (that is, how much lemonade does she need to make to break even)?

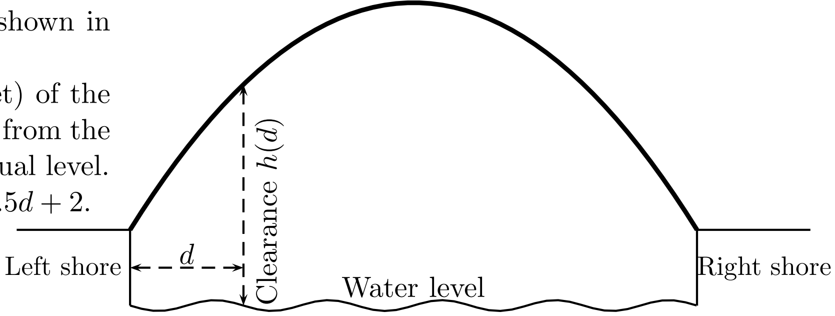
Solution: To break even, Marisa needs her revenue to be greater than or equal to her cost.

$$\begin{aligned} R(g) &\geq C(g) \\ 4g &\geq 15 + 3.5g \\ 0.5g &\geq 15 \\ g &\geq 30 \end{aligned}$$

So, in order to not lose money, Marisa must make at least 30 gallons of lemonade.

5. [16 points]

A bridge over the Huron River has the shape of a (symmetric) parabolic arch, as shown in the figure on the right.
Let $h(d)$ denote the clearance (in feet) of the bridge over a point in the river d feet from the left shore when the water is at its usual level. We are told that $h(d) = -0.07d^2 + 3.5d + 2$.



a. [8 points] By completing the square, find the maximum clearance of the bridge (that is, the clearance of the bridge at its highest point). *Remember to include units.*

Solution: We use the method of completing the square to find the vertex of the parabola that is the graph of the function $h(d)$.

$$\begin{aligned} h(d) &= -0.07d^2 + 3.5d + 2 \\ &= -0.07(d^2 - 50d) + 2 \\ &= -0.07(d^2 - 50d + (-25)^2 - (-25)^2) + 2 \\ &= -0.07((d-25)^2 - 625) + 2 \\ &= -0.07(d-25)^2 - 0.07(-625) + 2 \\ &= -0.07(d-25)^2 + 43.75 + 2 \\ &= -0.07(d-25)^2 + 45.75 \end{aligned}$$

Therefore, the vertex of the parabola is at the point $(25, 45.75)$. In terms of the bridge, the maximum clearance occurs at the vertex and is equal to 45.75 feet.

Answer: The maximum clearance of the bridge is 45.75 feet.

b. [3 points] At the bridge crossing, what is the width of the river (distance from left shore to right shore)? *Remember to include units.*

Solution: Since the bridge is symmetric, the highest point is at the point halfway across the river. From part (a), we know this point is at a distance of 25 feet from the left shore. Hence the distance from the left to the right shore is twice that amount, which is 50 feet.

Answer: The width of the river at the bridge crossing is 50 feet.

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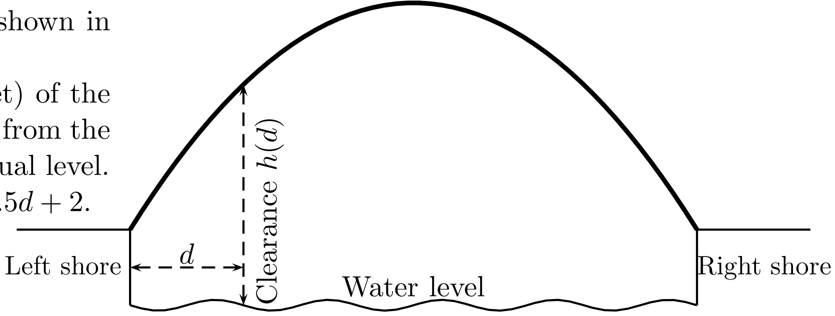
This is a continuation of the problem from the previous page. For your convenience, the original problem statement has been reprinted here.

A bridge over the Huron River has the shape of a (symmetric) parabolic arch, as shown in the figure on the right.
Let $h(d)$ denote the clearance (in feet) of the bridge over a point in the river d feet from the left shore when the water is at its usual level. We are told that $h(d) = -0.07d^2 + 3.5d + 2$.



c. [5 points] Murphy is rafting down the Huron river. As Murphy's raft is passing under the bridge, he decides to pull his raft over to the left shore of the river. Murphy is six feet tall. How close to the shore can he get before he hits his head on the bridge? (Assume that Murphy is standing upright, and that the height of the raft is negligible.)
Find an answer in *exact form* and then give an approximate value accurate to at least 2 decimal places.

Solution: We want to find d so that $h(d) = 6$. So we have $-0.07d^2 + 3.5d + 2 = 6$ or $-0.07d^2 + 3.5d - 4 = 0$. Applying the quadratic formula to this equation we find

$$\begin{aligned} d &= \frac{-3.5 \pm \sqrt{(3.5)^2 - 4(-0.07)(-4)}}{2(-0.07)} \\ &= \frac{-3.5 \pm \sqrt{12.25 - 1.12}}{-0.14} \\ &= \frac{-3.5 \pm \sqrt{11.13}}{-0.14} \end{aligned}$$

Note that both of these solutions are positive. However, since we want to know how close Murphy can get, we want the smaller of these two solutions. (The other gives the point that is as close to the right shore as possible.)
Hence Murphy can get to a distance of $\dfrac{-3.5 + \sqrt{11.13}}{-0.14}$ or approximately 1.17 feet of the left shore before he hits his head on the bridge.

6. [18 points] After a particularly rainy spring and early summer, the local mosquito population grew rapidly. A local group studying the mosquito population used traps to estimate the daily mosquito population. On the 15th day of their study, 750 mosquitoes were caught in these traps, and on the 32nd day, there were 6600 mosquitoes caught in these traps.

a. [8 points] Assuming the mosquito population grew at a constant percent rate for the first 32 days of the study, find a formula for $M(t)$, the number of mosquitoes caught in the traps on day t of the study for the first 32 days of the study. Any numbers appearing in your formula should either be in *exact form* or be *accurate to at least* 4 *decimal places.*

Solution: Since the population grew at a constant percent rate, the $M(t)$ is an exponential function. Thus $M(t)$ can be written as $M(t) = ab^t$ for some a and b. We find a and b using the data provided, i.e. that $M(15) = 750$ and $M(32) = 6600$.

We have $750 = ab^{15}$ and $6600 = ab^{32}$. Taking the ratio of these quantities, we find $\frac{6600}{750} = \frac{ab^{32}}{ab^{15}}$ so $8.8 = b^{17}$ and thus $b = (8.8)^{1/17} \approx 1.13647$.

To find a we use our computed value of b in one of the earlier equations: $750 = a(8.8^{1/17})^{15}$ so $750 = a(8.8^{15/17})$ and thus $a = 750(8.8^{-15/17}) \approx 110.0764$.

Answer: $\boxed{M(t) = 750(8.8^{-15/17})(8.8^{1/17})^t \text{ or } M(t) = 110.0764(1.1365)^t}$

b. [2 points] By what percent does the number of mosquitoes caught in the traps increase each day during the first 32 days of the study?
Your answer should be accurate to at least 2 decimal places.

Solution: Using the formula we found above, the daily growth factor is approximately 1.1365 so the daily growth rate is approximately $1.1365 - 1 = 0.1365$. Hence, the number of mosquitoes caught in the traps increased by $\boxed{\text{about } 13.65\%}$ each day during the first 32 days of the study.

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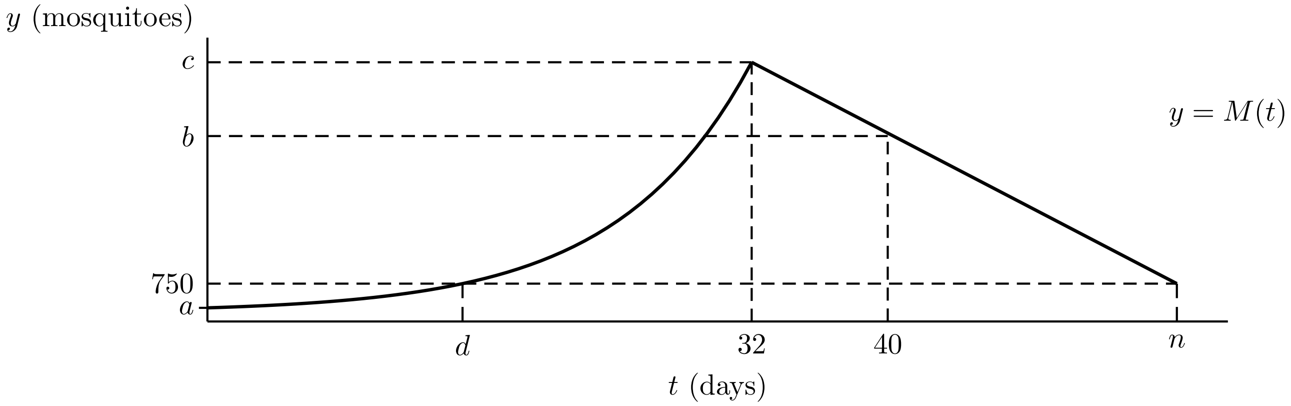
After reaching a peak of 6600, the number of mosquitoes caught in the traps each day began to decline at a constant rate, and on the 40th day of the study, the total number of mosquitoes caught in the traps was only 5560. The constant rate of decline continued until day n, when the total number of mosquitoes collected in the traps was back down to 750.
Recall that $M(t)$ is the number of mosquitoes caught in the traps on day t of the study. A graph of $M(t)$ is shown below. *Note that the graph is not drawn to scale.*



c. [4 points] Find the values of the constants a, b, c, and d shown in the graph above.

$a =$ 110.0764

$c =$ 6600

$b =$ 5560

$d =$ 15

d. [4 points] Find the value of n.

Solution: Since the number of mosquitoes caught declines at a constant rate beginning on day 32, $M(t)$ is *linear* for $32 < t \leq n$.
The average rate of change of M from $t = 32$ to $t = 40$ is

$$\frac{\Delta M}{\Delta t} = \frac{M(40) - M(32)}{40 - 32} = \frac{5560 - 6600}{8} = \frac{-1040}{8} = -130 \text{ mosquitoes per day.}$$

Thus, using point-slope form, a formula for $M(t)$ for $32 < t \leq n$ is $M(t) = 6600 - 130(t - 32)$.
Since $M(n) = 750$, we solve $750 = 6600 - 130(n - 32)$ for n.

$$\begin{aligned} 750 &= 6600 - 130(n - 32) \\ -5850 &= -130(n - 32) \\ 45 &= n - 32 \\ 77 &= n \end{aligned}$$

Answer: $n =$ 77

7. [10 points] In this problem, we consider two functions:

- $W(s)$ is the wind chill[1] (in degrees Fahrenheit) when the temperature is 30 degrees Fahrenheit and the wind speed is s mph (miles per hour).
- $B(c)$ is the time (in minutes) it takes to develop frostbite on exposed skin when the wind chill is c degrees Fahrenheit.

Assume both W and B are invertible. Fill in each blank below with one of the possible answers given below. Note that a given answer may be used in more than one blank, and that not all possible answers will be used.

Possible Answers:

20	$W(20)$	$B(20)$	W(20)+B(20)
$W^{-1}(20)$	$B^{-1}(20)$	$W(B(20))$	$B(W(20))$
$W^{-1}(B^{-1}(20))$	$B^{-1}(W^{-1}(20))$	$W(B^{-1}(20))$	$B(W^{-1}(20))$

Assume throughout this problem that the temperature is 30 degrees Fahrenheit.

a. [2 points] If the wind chill is $\underline{W(20)}$ degrees Fahrenheit, the wind speed is 20 mph.

b. [2 points] When the wind speed is 20 mph, exposed skin will develop frostbite in $\underline{B(W(20))}$ minutes.

c. [2 points] If the wind chill is 20 degrees Fahrenheit, then the wind speed is $\underline{W^{-1}(20)}$ mph.

d. [2 points] If the wind chill is 20 degrees Fahrenheit, then it will take exposed skin

$\underline{B(20)}$ minutes to develop frostbite.

e. [2 points] When the wind chill is $B^{-1}(20)$ degrees Fahrenheit, exposed skin will develop

frostbite in $\underline{20}$ minutes.

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[1] Note that *wind chill* is the temperature it "feels like".