Math 105 — Second Midterm

November 14, 2011

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	12	
3	10	
4	13	
5	13	
6	10	
7	14	
8	14	
Total	100	

1. [14 points] No work or explanation is expected on this page.

a. [4 points] If the graph of the function $y = k(w)$ has a vertical asymptote $w = 5$ and a horizontal asymptote $y = 3$, then what, if any, asymptotes does the graph of $y = 2k(4w) + 3$ have? (Write "None" in the blank if no asymptotes of that type can be found from the information provided.)

Solution: The graph of $y = 2k(4w) + 3$ is obtained from the graph of $y = k(w)$ by first compressing horizontally by a factor of $1/4$ then stretching vertically by a factor of 2, and finally shifting the graph up by 3 units. These transformations send the vertical asymptote $w = 5$ to $w = 5/4$ and the horizontal asymptote $y = 3$ to $y = 9$.

Vertical: $w = 5/4$ **Horizontal:** $y = 9$

b. [2 points] If the function $f(q)$ has a zero $q = -2$, find a zero of the function $4f(3(q - 1))$.

Solution: The graph of $4f(3(q - 1))$ is obtained from the graph of $f(q)$ by first compressing horizontally by a factor of $1/3$ (which sends $q = -2$ to $q = -2/3$), then shifting right by 1 unit (taking $q = -2/3$ to $q = 1/3$) and finally stretching vertically by a factor of 4 (which does not change the horizontal intercepts). Hence, $q = 1/3$ is a zero of $4f(3(q - 1))$.
An alternate approach: $4f(3(q - 1)) = 0$ if $f(3(q - 1)) = 0$ so since $f(-2) = 0$, the transformed function has a zero when $3(q - 1) = -2$. Solving this equation, we find $q = 1/3$.

Answer: $q = 1/3$

c. [2 points] Is the function $p(t) = \sin(t) + \pi$ even, odd, or neither? *Circle* ONE *answer.*

Solution: The graph of $y = \sin(t)$ is symmetric about the origin as sine is an odd function. However, the graph of $\sin(t) + \pi$, which is the graph of $\sin(t)$ shifted up π units, is not symmetric about the origin. It is also not symmetric about the vertical axis. Therefore the function $\sin(t) + \pi$ is neither even nor odd.

even **odd** $\boxed{\textbf{neither even nor odd}}$

d. [2 points] Is there a number whose natural log is $8,675,309$? *Circle* ONE *answer.*

Solution: The natural log of $e^{8,675,309}$ is $8,675,309$. (The range of the natural logarithm function is all real numbers.)

$\boxed{\textbf{yes}}$ no

e. [4 points] Suppose $\cos(\phi) = -0.6$ and $\pi \le \phi \le 2\pi$. Find $\sin(\phi)$ and $\tan(\phi)$.

Solution: By the Pythagorean Identity, $\sin^2(\phi) + \cos^2(\phi) = 1$, so $\sin^2(\phi) + (-0.6)^2 = 1$. Solving this equation, we find $\sin^2(\phi) = 0.64$ so $\sin(\phi) = \pm\sqrt{0.64} = \pm 0.8$. Since $\pi \le \phi \le 2\pi$, the sign of $\sin(\phi)$ is negative. Hence $\sin(\phi) = -0.8$.
By definition, we then have $\tan(\phi) = \dfrac{\sin(\phi)}{\cos(\phi)} = \dfrac{-0.8}{-0.6} = \dfrac{4}{3}$.

Answers: $\sin(\phi) =$ -0.8 $\tan(\phi) =$ $4/3$

2. [12 points] For each equation below, solve EXACTLY for the specified variable. Show your work carefully and write your final answer on the answer blank provided.

a. [4 points] If $5e^{2t+3} = 6$, solve for t.

Solution: Taking the natural log of both sides of the equation, we get $\ln\left(5e^{2t+3}\right) = \ln(6)$. Using the properties of ln, we find $\ln(5) + (2t+3) = \ln(6)$. We can rearrange to get $2t = \ln(6) - \ln(5) - 3$, so that $t = \dfrac{\ln(6) - \ln(5) - 3}{2}$.

Answer: The exact value of t is $\dfrac{\ln(6) - \ln(5) - 3}{2}$ **or** $\dfrac{\ln(6/5) - 3}{2}$.

b. [4 points] If $\ln(3^x) = \ln(2^x) + 7$, solve for x.

Solution: Using the properties of the natural logarithm, this equation can be written $x\ln(3) = x\ln(2) + 7$. Collecting the x terms on one side of the equation, we find that $(\ln(3) - \ln(2))x = 7$, so that $x = \dfrac{7}{\ln(3) - \ln(2)}$.

Answer: The exact value of x is $\dfrac{7}{\ln(3) - \ln(2)}$ **or** $\dfrac{7}{\ln(3/2)}$.

c. [4 points] If $\log(w+1) - \log(2-w) = 2$, solve for w.

Solution: Using the properties of log, we can rewrite this equation as $\log\left(\frac{w+1}{2-w}\right) = 2$. By the definition of the logarithm (or raising 10 to the power of both sides), we get $\dfrac{w+1}{2-w} = 10^2 = 100$. Clearing the denominator, we find $w + 1 = 100(2 - w)$. Collecting all of the w terms on one side, we get $101w = 199$, so $w = \dfrac{199}{101}$.

Answer: The exact value of w is $\dfrac{199}{101}$.

3. [10 points] Below, you are given a table with some data about two functions, $f(x)$ and $g(x)$:

x	0	1	2
$f(x)$	2	-4	3
$g(x)$	0	5	-1

In addition to this table, we are told the following facts:

- $f(x)$ is an even function.
- $g(x)$ is an odd function.
- There is a third function $h(x)$. The graph of $y = h(x)$ is obtained by shifting the graph of $y = f(x)$ one unit to the right.

Use this information to answer the questions below.

a. [2 points] Compute $h(1)$.

Solution: Since the graph of $y = h(x)$ is obtained by shifting the graph of $y = f(x)$ one unit to the right, we have $h(x) = f(x - 1)$. So, $h(1) = f(1 - 1) = f(0) = 2$.
Alternatively, shifting the graph of $y = f(x)$ one unit to the right takes the point $(0, 2)$ to the point $(1, 2)$. So $(1, 2)$ is a point on the graph of $y = h(x)$ and thus $h(1) = 2$.

Answer: $h(1) =$ 2 .

b. [2 points] Compute $f(-1)$.

Solution: We are told $f(x)$ is even, so $f(x) = f(-x)$. Therefore, $f(-1) = f(1) = -4$.

Answer: $f(-1) =$ -4 .

c. [2 points] Compute $g(2) + g(-2)$.

Solution: We are told $g(x)$ is odd, so $g(-x) = -g(x)$. Therefore, $g(-2) = -g(2) = 1$, so $g(2) + g(-2) = -1 + 1 = 0$.

Answer: $g(2) + g(-2) =$ 0 .

d. [2 points] If $j(x) = f(2x) + 1$, compute the value of $j(1)$.

Solution: We have $j(1) = f(2 \cdot 1) + 1 = f(2) + 1 = 3 + 1 = 4$.

Answer: $j(1) =$ 4 .

e. [2 points] If $g(x) = k(-2x)$, compute the value of $k(-4)$.

Solution: The relationship between $g(x)$ and $k(x)$ shows us that the graph of $y = g(x)$ can be obtained from the graph of $y = k(x)$ by compressing horizontally by a factor of $\frac{1}{2}$ and reflecting across the vertical axis. From this, we can see that $g(2) = k(-4)$. From the table, we know $g(2) = -1$, so $k(-4) = -1$.
Alternatively, since $k(-2x) = g(x)$, we see that $k(-4) = k(-2 \cdot 2) = g(2) = -1$.

Answer: $k(-4) =$ -1 .

4. [13 points] *For each problem on this page, show your work step-by-step.*
(Don't forget to use appropriate units in your answers.)

Ozone is a molecule consisting of three oxygen atoms that is unstable and decays to the stable form of oxygen. The half-life of gaseous ozone at a temperature of 20°C is 3 days (72 hours).

a. [4 points] Find the continuous hourly percent decay rate of gaseous ozone at 20°C. *Give your answer in exact form.*

Solution: If the initial amount of gaseous ozone is a and the continuous hourly percent decay rate of gaseous ozone is k, then the amount of ozone after t hours is given by ae^{kt}. Since the half-life is given as 72 hours, then after 72 hours, the amount of ozone is $0.5a$, i.e. we have the equation $0.5a = ae^{72k}$. Then $0.5 = e^{72k}$, so, by the definition of ln (or taking the natural logarithm of both sides) we have $\ln(0.5) = 72k$ so $k = \dfrac{\ln(0.5)}{72}$.
Note: $\frac{\ln(0.5)}{72} \approx -0.009627$, so it decays at a continuous rate of about 0.9627% per hour.

Answer: $\dfrac{\ln(0.5)}{72}$

b. [4 points] At a temperature of 20°C, how long does it take for the amount of gaseous ozone to be reduced by 90%? *Give your answer in exact form.*

Solution: The amount of gaseous ozone remaining from an initial quantity a after t hours is ae^{kt} where $k = \ln(0.5)/72$ (as found in part (a)). When the amount has been reduced by 90%, the amount left is $0.1a$, so we have $0.1a = ae^{kt}$. Then $0.1 = e^{kt}$ so by the definition of ln (or taking the natural log of both sides), we have $\ln(0.1) = kt$ so $t = \ln(0.1)/k$. Using our value of k from part (a), this is $t = \dfrac{\ln(0.1)}{\ln(0.5)/72} = \dfrac{72\ln(0.1)}{\ln(0.5)}$.
Note: $\frac{72\ln(0.1)}{\ln(0.5)} \approx 239.1788$, so it takes approximately 239.2 hours for gaseous ozone to be reduced by 90%.

Answer: $\dfrac{72\ln(0.1)}{\ln(0.5)}$ **hours**

When ozone is dissolved in water, it is referred to as "aqueous ozone." At a temperature of 20°C, aqueous ozone decays at a rate of 12.5% per hour.

c. [5 points] Suppose that in a 20°C lab, the amount of aqueous ozone is initially 5 times the amount of gaseous ozone. When will the two amounts be equal?
Give your answer in exact form.

Solution: Let g be the initial amount of gaseous ozone. Then the initial amount of aqueous ozone is $5g$, and, since its decay rate is 12.5% per hour, the amount of aqueous ozone remaining after t hours is $(5g)(0.875)^t$. From part (a) above, the amount of gaseous ozone remaining after t hours is $ge^{t\ln(0.5)/72}$. So, we are to find the value of t when these two amounts are equal, i.e. to solve the equation $ge^{t\ln(0.5)/72} = (5g)(0.875)^t$. Dividing both sides by g gives us $e^{t\ln(0.5)/72} = (5)(0.875)^t$. Taking the natural log of both sides of this equation, we then find $\ln\left(e^{t\ln(0.5)/72}\right) = \ln(5(0.875)^t)$ so $t\ln(0.5)/72 = \ln(5) + t\ln(0.875)$. Collecting like terms and factoring out t then gives us $t(\ln(0.5)/72 - \ln(0.875)) = \ln(5)$ so $t = \dfrac{\ln(5)}{\ln(0.5)/72 - \ln(0.875)}$. (Note that this is approximately 13 hours.)

Answer: $\dfrac{\ln(5)}{\ln(0.5)/72 - \ln(0.875)}$ **hours**

5. [13 points] In Western Japan (as in many other places around the world), electrical outlets supply power in the form of alternating current, which means that the voltage changes over time. The voltage goes from a maximum of 141 volts to a minimum of -141 volts and back again, 60 times every second. Let $V(t)$ be the voltage of an electrical outlet in Western Japan, where t is time, in seconds. Assume that $V(t)$ is obtained from the function $\sin(t)$ by performing shifts, stretches and/or reflections, and that at time $t = 0$, the voltage of the outlet is at 141 volts.

a. [5 points] Find the period, amplitude, and midline of $V(t)$. (Include units.)

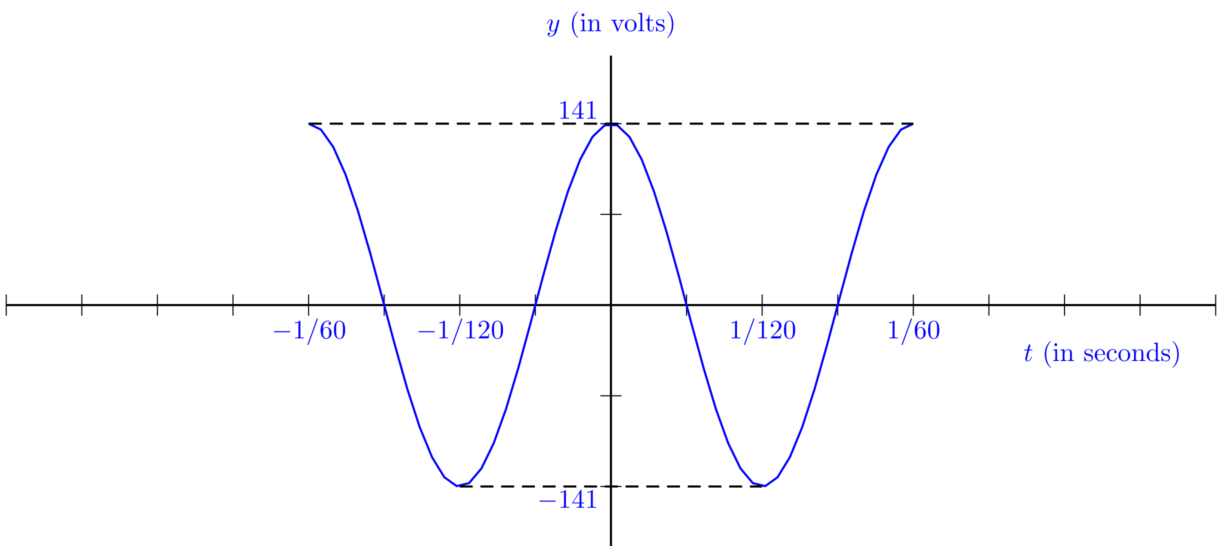
Solution:

Period: $1/60$ **of a second**

Amplitude: 141 **volts**

Midline: $y = 0$ **volts**

b. [5 points] On the axes provided, sketch a graph of $y = V(t)$ for two periods.
(*Clearly label the axes and be very careful with the shape and key features of your graph.*)



c. [3 points] The power, $P(t)$ (in Watts) dissipated by a particular electric lightbulb also varies with time. The graph of $y = P(t)$ is obtained from the graph of $y = V(t)$ by performing, in order, the following transformations:

1. A horizontal compression by a factor of $\frac{1}{2}$
2. A vertical compression by a factor of $\frac{60}{141}$
3. A shift upward by 60 units

Find a formula for $P(t)$ in terms of $V(t)$.

Solution: Horizontally compressing the graph of $y = V(t)$ by a factor of $1/2$ gives the graph of $y = V(2t)$. Vertically compressing this by a factor of $60/141$ gives the graph of $y = \dfrac{60}{141}V(2t)$, and finally shifting up by 60 units gives the graph of $y = \dfrac{60}{141}V(2t) + 60$

Answer: $P(t) = \dfrac{60}{141}V(2t) + 60$

6. [10 points] Mr. and Mrs. Johnson have 4 children: Alana, Brentley, Clarissa, and Donovan. Let $A(t)$, $B(t)$, $C(t)$, and $D(t)$ denote the height, in inches, of Alana, Brentley, Clarissa, and Donovan, respectively, at time t, measured in years since January 1, 1990. Alana was born on January 1, 1990.

a. [3 points] Alana and Brentley are twins (i.e. they were born at the same time), but Brentley is shorter. He is always 5% shorter than Alana. Write a formula for $B(t)$ in terms of $A(t)$.

Solution: Brentley is 5% shorter than Alana, so his height is 0.95 times the height of Alana. This means that $B(t) = 0.95A(t)$.

Answer: $B(t) =$ $0.95A(t)$

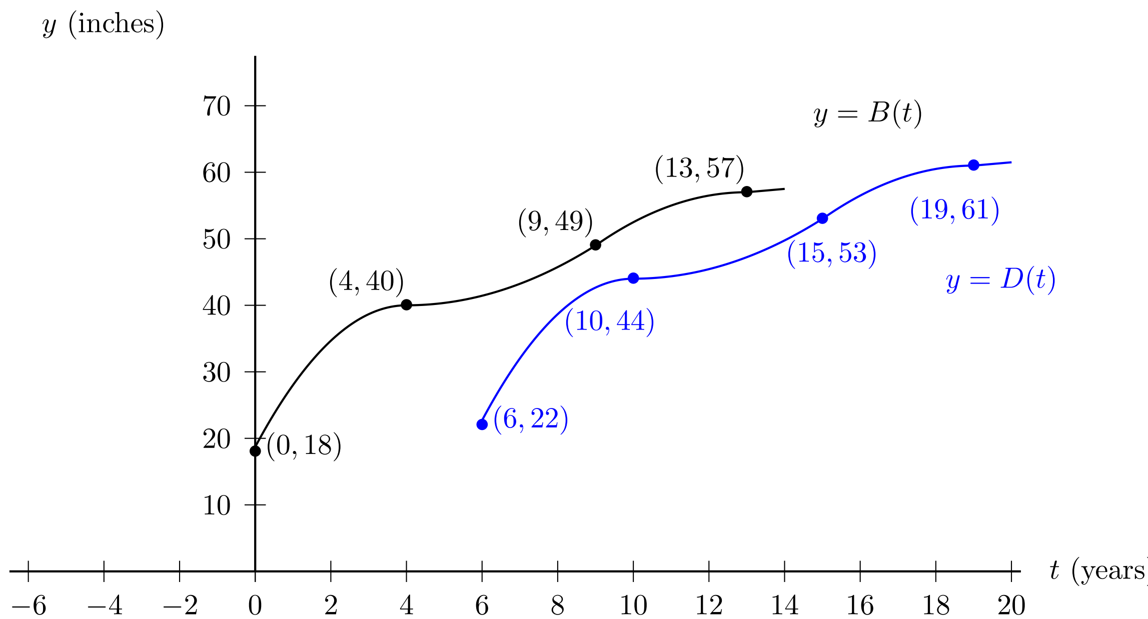
b. [3 points] Clarissa was born exactly 4 years after Alana. Clarissa is always the same height as Alana was when she was the same age. Write a formula for $C(t)$ in terms of $A(t)$.

Solution: In year t, Clarissa is the same age as Alana was 4 years earlier, in year $t-4$. Since Clarissa is the same height as Alana was when she was the same age, we find $C(t) = A(t-4)$.

Answer: $C(t) =$ $A(t-4)$

c. [4 points] Donovan was born exactly 6 years after Brentley. However, Donovan has a larger build, and is always 4 inches taller than Brentley was at the same age. Below, you are given a portion of the graph of $y = B(t)$. The coordinates of four points on the graph are labeled. Using this information, sketch as much as possible of the graph of $y = D(t)$ on the same axes. **Label four points on your graph**.

Solution: In year t, Donovan is the same age as Brentley was 6 years earlier, in year $t-6$. Donovan is 4 inches taller than Brentley was at the same age, so we have $D(t) = B(t-6) + 4$. This means that the graph of $y = D(t)$ is obtained from the graph of $y = B(t)$ by shifting 6 units to the right and 4 units upwards.



7. [14 points] In music, the *pitch* of a tone, P, measured in *cents*, is a function of the tone's frequency, f, measured in hertz. The pitch is defined to be

$$P = 6000 + k \ln\left(\frac{f}{f_0}\right)$$

where f_0 is the frequency of a tone called "middle C", and k is a constant.

a. [2 points] What is the pitch of "middle C"? (*Remember to include units.*)

Solution: The frequency of "middle C" is f_0, so the pitch of "middle C" is $6000 + k\ln\left(\frac{f_0}{f_0}\right) = 6000 + k\ln(1) = 6000 + k(0) = 6000$.

Answer: 6000 **cents**

b. [3 points] If the frequency of one tone is two times the frequency of middle C, then the pitch of that tone is 7200 cents. Use this information to find the *exact* value of k. Then give an approximation of k rounded to the nearest 0.1.

Solution: If the frequency of a tone is two times the frequency f_0 of middle C, then its frequency is $2f_0$. So we have $7200 = 6000 + k\ln\left(\frac{2f_0}{f_0}\right) = 6000 + k\ln(2)$. Solving for k we find $1200 = k\ln(2)$ so $k = 1200/\ln(2) \approx 1731.2$.

Exact value of k: $\dfrac{1200}{\ln 2}$ **Approximation:** 1731.2

Use the *approximation* of k you found in part (b) to answer the questions below. (If you were unable to answer part (b), leave your answers below in terms of k.)

c. [4 points] Let P_1 and P_2 represent the pitches of tones of frequency f_1 and f_2, respectively. Find a formula for the difference in pitches, $P_2 - P_1$, in terms of the two frequencies f_1 and f_2. *Simplify your answer; your formula should* **not** *involve* f_0.

Solution: We have

$$\begin{aligned} P_2 - P_1 &= \left(6000 + k\ln\left(\frac{f_2}{f_0}\right)\right) - \left(6000 + k\ln\left(\frac{f_1}{f_0}\right)\right) = k\ln\left(\frac{f_2}{f_0}\right) - k\ln\left(\frac{f_1}{f_0}\right) \\ &= k\left(\ln\left(\frac{f_2}{f_0}\right) - \ln\left(\frac{f_1}{f_0}\right)\right) = k\ln\left(\frac{f_2/f_0}{f_1/f_0}\right) = k\ln\left(\frac{f_2}{f_1}\right) \end{aligned}$$

Using our approximation of k from above, we find that $P_2 - P_1$ is approximately $1731.2\ln\left(\frac{f_2}{f_1}\right)$.

Answer: $P_2 - P_1 = k\ln\left(\frac{f_2}{f_1}\right) \approx 1731.2\ln\left(\frac{f_2}{f_1}\right)$

d. [5 points] The tone called "A above middle C" has a frequency of 440 hertz. Find the frequency of the tone whose pitch is 400 cents higher than the pitch of "A above middle C." (*Remember to include units.*)

Solution: Let f denote the frequency of the tone whose pitch is 400 cents higher than "A above middle C." Using the solution to part (c), we know that $400 = k\ln\left(\frac{f}{440}\right)$. Dividing through by k gives $\ln\left(\frac{f}{440}\right) = 400/k$. Exponentiating both sides gives $\dfrac{f}{440} = e^{400/k}$. We then find that $f = 440e^{400/k} \approx 554.37$ Hz.

Answer: **About** 554.37 **Hz**

8. [14 points] There is a bicycle wheel surrounded by a tire of uniform thickness. The wheel itself is 33 centimeters in radius, and the tire is 4 centimeters thick. The wheel has seven evenly-spaced spokes, one of which is initially pointing straight to the right. (See diagram below.)

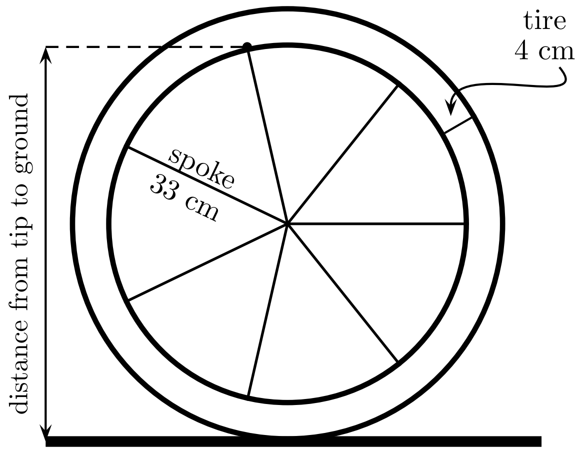


Figure not drawn to scale

a. [2 points] What is the exact angle (in radians) between two adjacent spokes?

Solution: Since there are 2π radians in a complete circle, the angle between two adjacent spokes is $\frac{2\pi}{7}$ radians.

Answer: $\frac{2\pi}{7}$ **radians**

b. [4 points] Find the distance from the tip of the highest spoke to the ground. (*This distance is labeled as "distance from tip to ground" in the diagram above.*)

Solution: Since the angle between two adjacent spokes is $\frac{2\pi}{7}$, the highest spoke makes an angle of $2(\frac{2\pi}{7}) = \frac{4\pi}{7}$ with the rightward-pointing spoke. The radius of the wheel is 33 centimeters, so the tip of the highest spoke is $33\sin\left(\frac{4\pi}{7}\right)$ centimeters above the center of the wheel. The center of the wheel is 33 centimeters above the bottom of the wheel, which is 4 centimeters above the ground, so in total, the distance from the tip of the highest spoke to the ground is $33\sin\left(\frac{4\pi}{7}\right) + 33 + 4 \approx 69.1726$ centimeters.

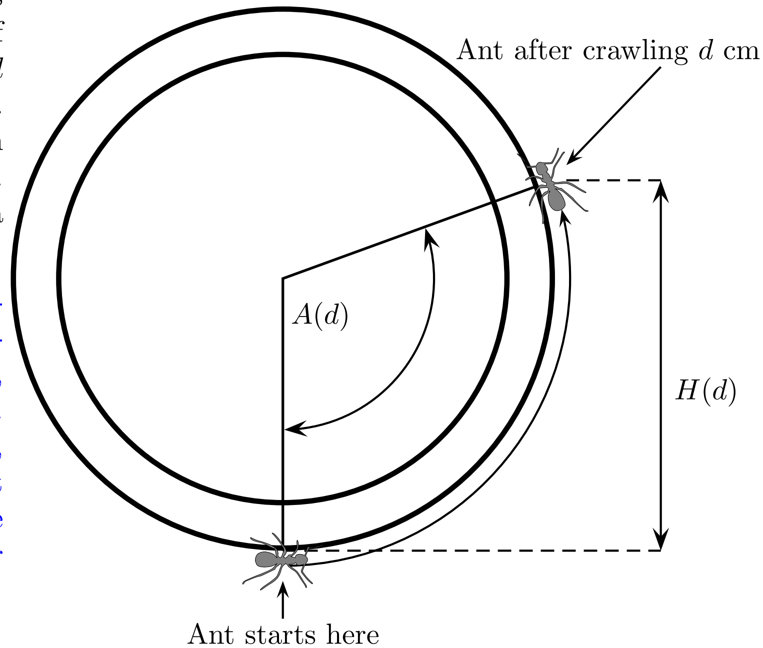
Answer: $33\sin\left(\frac{4\pi}{7}\right) + 37 \approx 69.1726$ **centimeters**

This problem continues on the next page.

This is a continuation of the problems from the previous page.

Recall: There is a bicycle wheel surrounded by a tire of uniform thickness. The wheel itself is 33 centimeters in radius, and the tire is 4 centimeters thick.

c. [4 points] One day, while the bicycle is parked, an ant crawls onto the bottom of the tire. The ant crawls for a distance of d centimeters along the outside of the tire. Let $A(d)$ denote the angle, measured in radians, through which the ant crawled. (See diagram on right.) Find a formula for $A(d)$ in terms of d.



Solution: The relationship between angle measurement A in radians and arc length d is $d = Ar$, where r is the radius of the circle. The ant is on the outside of the tire, so the radius of the circle of interest is $33 + 4 = 37$ centimeters. So, we get $d = A(d) \cdot 37$. We can solve for $A(d)$ to get $A(d) = \frac{d}{37}$.

Answer: $A(d) = \frac{d}{37}$

d. [4 points] The ant from part (c), after crawling through a distance of d centimeters, drops off of the tire and falls to the ground. Let $H(d)$ denote the distance, in centimeters, that the ant falls. (See diagram above.) Find a formula for $H(d)$ in terms of d.

Solution: The angle between the bottom of the wheel and the rightward-pointing horizontal is a quarter-circle, or $\frac{2\pi}{4} = \frac{\pi}{2}$ radians. The ant has crawled through an angle of $A(d)$ from the bottom of the wheel, so the ant's position makes an angle of $A(d) - \frac{\pi}{2} = \frac{d}{37} - \frac{\pi}{2}$ with the horizontal. At this point, the ant is a distance of $37 \sin\left(\frac{d}{37} - \frac{\pi}{2}\right)$ centimeters above the center of the wheel, so the distance the ant falls in total is $H(d) = 37 + 37\sin\left(\frac{d}{37} - \frac{\pi}{2}\right)$ centimeters.

Answer: $H(d) = 37 + 37\sin\left(\frac{d}{37} - \frac{\pi}{2}\right)$