Math 105 — Final Exam

December 15, 2011

Name: EXAM SOLUTIONS

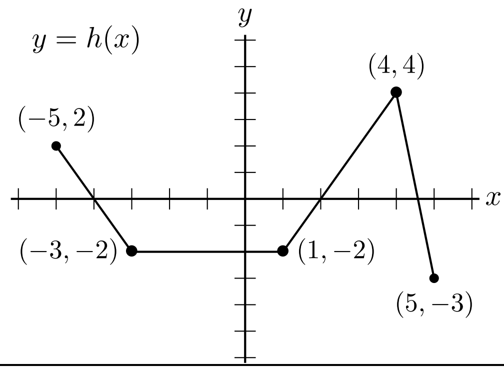
Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	9	
2	10	
3	11	
4	12	
5	10	
6	12	
7	8	
8	12	
9	7	
10	9	
Total	100	

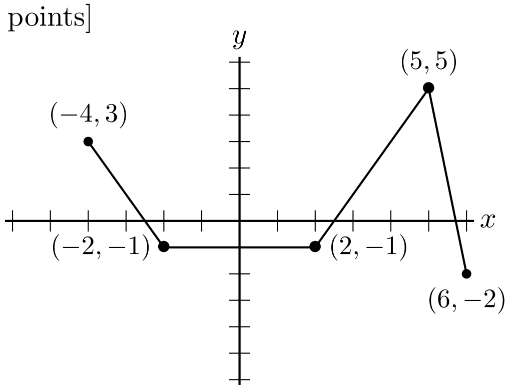
1. [9 points]

The graph of a function $h(x)$ is shown on the right. Below are the graphs of several transformations of $h(x)$. For each of these graphs, write the letter of the ONE function from the list on the right of the page whose graph is shown. (**Clearly** write the capital letter of your choice on the answer blank provided.) *No work or explanation is required.*



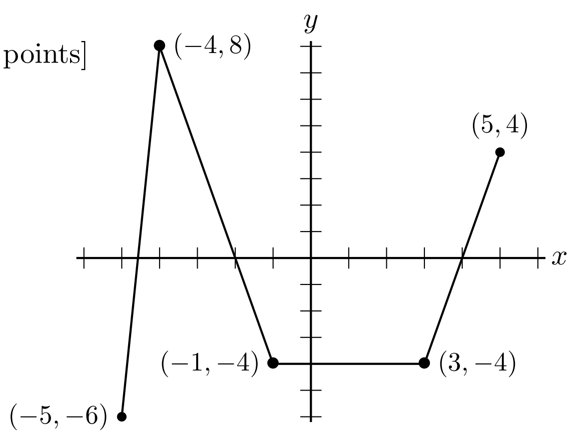
a. [3 points]

Answer: B



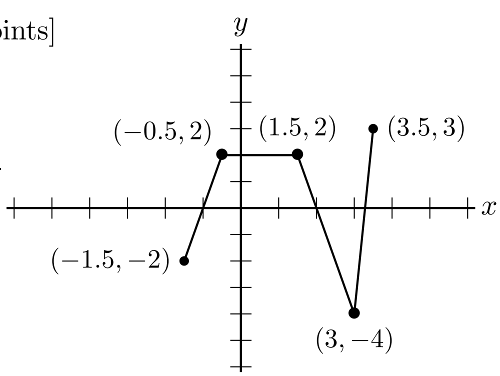
b. [3 points]

Answer: N



c. [3 points]

Answer: V



Answer Choices

A. $h(x+1)+1$

B. $h(x-1)+1$

C. $h(x+1)-1$

D. $h(x-1)-1$

E. $h(-x)+1$

F. $h(-x)-1$

G. $-h(x)+1$

H. $-h(x)-1$

I. $-h(x+1)$

J. $-h(x-1)$

K. $h(-x)$

L. $-h(-x)$

M. $2h(x)$

N. $2h(-x)$

O. $-2h(x)$

P. $\frac{1}{2}h(x)$

Q. $\frac{1}{2}h(-x)$

R. $-\frac{1}{2}h(x)-1$

S. $\frac{1}{2}h(x-1)$

T. $h(-2(x-1))$

U. $-h(2x-1)$

V. $-h(2(x-1))$

W. $-h(\frac{1}{2}x-1)$

X. $h(-\frac{1}{2}(x+1))$

Y. $-h(\frac{1}{2}(x-1))$

Z. NONE OF THESE

2. [10 points] A movie theater is considering selling discount tickets for opening night of a new vampire movie. The management estimates that they will sell 1100 tickets if they set the price of tickets at \$7 each. However, if they charge \$10 for each ticket, the theater will only sell 800 tickets. Let $T(p)$ be the number of tickets the theater will sell if the price of each ticket is p dollars. Assume that $T(p)$ is a linear function.

a. [4 points] Find a formula for $T(p)$ in terms of p.

Solution: The slope (average rate of change) is

$$\frac{T(10) - T(7)}{10 - 7} = \frac{800 - 1100}{10 - 7} = -100$$

tickets per dollar. Using point-slope form, we find $T(p) = 800 - 100(p - 10) = 1800 - 100p$.

$T(p) =$ $1800 - 100p$

b. [1 point] Let $R(p)$ be the total amount of money the theater takes in from ticket sales if the price of each ticket is p dollars. Find a formula for $R(p)$ in terms of p.

Solution: If $T(p)$ tickets are sold at a price of p dollars each, then the total amount of money taken in from ticket sales is $R(p) = p(T(p)) = p(1800 - 100p) = 1800p - 100p^2$.

$R(p) =$ $1800p - 100p^2$

c. [5 points] By completing the square, put $R(p)$ in vertex form. *Show step by step work.* How much should the theater charge for each ticket if they want to maximize the amount of money they take in? How much would the theater take in if they charged this amount?

Solution: Using the method of completing the square, we find

$$\begin{aligned} R(p) &= -100p^2 + 1800p = -100(p^2 - 18p) \\ &= -100(p^2 - 18p + 81 - 81) \\ &= -100\left((p-9)^2 - 81\right) = -100(p-9)^2 + 8100. \end{aligned}$$

Hence the vertex of the graph of $R(p)$ is $(9, 8100)$. Since the leading coefficient is negative, the parabola opens downward and the vertex gives a maximum. Therefore, the maximum of $R(p)$ is 8100 and occurs at $p = 9$, i.e. the maximum revenue of the theater would be \$8100 at a ticket price of \$9.

Vertex form: $R(p) =$ $-100(p-9)^2 + 8100$

Ticket price: \$9 **Money taken in:** \$8100

3. [11 points] *No work or explanation is required on this page.*

a. [4 points] Determine which, if any, of the functions listed below satisfy ALL of the following:

- It has a zero at $x = -5$.
- Its long-run behavior satisfies $y \to -\infty$ as $x \to \infty$.
- Its long-run behavior satisfies $y \to -\infty$ as $x \to -\infty$.

(Circle all of the functions that satisfy all three conditions, if there are any; otherwise, circle NONE OF THESE.)

i. $y = -4(x-5)(x-1)^2(x+2)$

ii. $y = 2(x+5)(x+1)^2(x-2)^2$

iii. $y = -4(x+5)(x+1)^2(x-2)$

iv. $y = \dfrac{-4(x-5)(x+1)}{x+5}$

v. $y = \dfrac{-4(x+5)(x+1)^2(x-5)}{x^2+25}$

vi. $y = \dfrac{-2(x+5)(x-5)(x-2)}{x^2+25}$

vii. NONE OF THESE

b. [3 points] Which, if any, of the following functions have $y = 2$ as a horizontal asymptote? *Circle your answer(s).*

i. $y = \dfrac{6x^4 - 5x^2 + 3}{3x^4 + 2x - 1}$

ii. $y = \dfrac{(2x-1)(x+3)(x-5)}{(x+1)(x-4)}$

iii. $y = \dfrac{2e^x + x^2}{2 + e^x}$

iv. $y = \dfrac{2\ln x + x}{\ln x + 3}$

v. NONE OF THESE

c. [4 points] Data for a function $g(s)$ is given in the following table.

s	-4	-2	-1	1	3
$g(s)$	13	5	2	-2	-4

For each property listed below, determine whether $g(s)$ could have that property on the entire domain $[-4, 3]$. (Circle each term that *could* describe $g(s)$, if there are any; otherwise, circle NONE OF THESE.)

i. INCREASING

ii. DECREASING

iii. CONCAVE UP

iv. CONCAVE DOWN

v. AN ODD FUNCTION

vi. AN EVEN FUNCTION

vii. AN INVERTIBLE FUNCTION

viii. A LINEAR FUNCTION

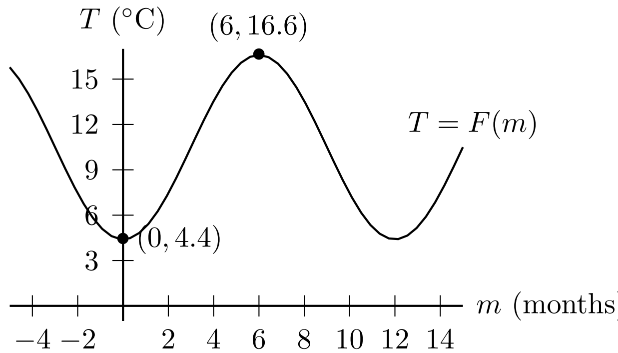
ix. AN EXPONENTIAL FUNCTION

x. NONE OF THESE

4. [12 points]

a. [7 points]

The average temperature, T, in degrees Celsius, for the city of Forks, Washington, can be modeled by the sinusoidal function $F(m)$, where m is measured in months after January 1 (so $m = 0$ represents January 1). A portion of the graph of $T = F(m)$ is shown on the right.



Find the period, amplitude, midline, and a formula for the sinusoidal function $F(m)$ shown above. (*Include units for the period and amplitude.*)

Period: 12 **months**

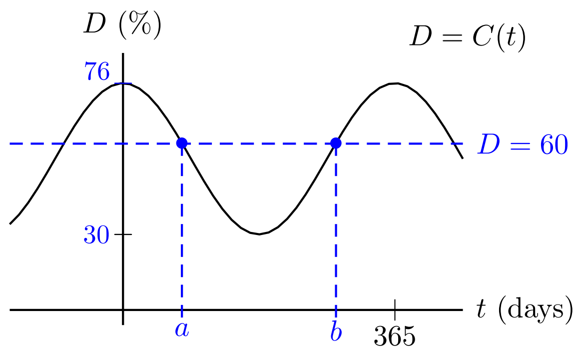
Amplitude: 6.1°**C**

Midline: $T = 10.5$

Formula: $F(m) =$ $-6.1\cos(\frac{\pi}{6}m) + 10.5$ **(many possible answers)**

b. [5 points]

Suppose the chance of significant cloud cover in Seattle on day t of the year is D%. D can be approximated by the function $C(t) = 23\cos(0.0172t) + 53$. A portion of the graph of $D = C(t)$ is shown to the right. A family in Forks wants to visit Seattle when the chance of significant cloud cover is at least 60%. Find ALL solutions to the equation $C(t) = 60$ for $0 \le t \le 365$.



For full credit, you should solve this problem algebraically and show each step clearly. Your answer(s) should either be in exact form or be accurate to at least 2 decimal places.

Solution: Using the line $D = 60$ in the graph above, we see that there are two solutions for $0 \le t \le 365$ (labeled a and b above). By symmetry, we see that $b = 365 - a$. Solving the equation $23\cos(0.0172t) + 53 = 60$ we find $23\cos(0.0172t) = 7$ so $\cos(0.0172t) = \frac{7}{23}$. So *one* solution is given by $0.0172t = \cos^{-1}(\frac{7}{23})$, i.e. $t = \frac{\cos^{-1}(\frac{7}{23})}{0.0172} \approx 73.346$. This is the solution a shown in the graph. So, the two solutions are $a = \frac{\cos^{-1}(\frac{7}{23})}{0.0172} \approx 73.346$ and $b = 365 - \frac{\cos^{-1}(\frac{7}{23})}{0.0172} \approx 291.654$.

Answer(s): $\dfrac{\cos^{-1}(\frac{7}{23})}{0.0172}(\approx 73.346)$ **and** $365 - \dfrac{\cos^{-1}(\frac{7}{23})}{0.0172}(\approx 291.654)$

5. [10 points]

a. [2 points] Consider the function $P(t)$ defined by

$$P(t) = \begin{cases} \frac{70t(t-6)}{(t-10)(t+2)} & \text{if } 0 \le t \le 5 \\ 2 + 5e^{5-t} & \text{if } t > 5. \end{cases}$$

Evaluate $P(5)$ and $P(P(5))$.

Solution: Since $5 \le 5$, we use the first part of the formula to find $P(5)$. So $P(5) = \frac{70(5)(5-6)}{(5-10)(5+2)} = 10$. Then $P(P(5)) = P(10) = 2 + 5e^{5-10} = 2 + 5e^{-5}$. (Note that we use the second part of the formula for $P(t)$ to compute $P(10)$ since $10 > 5$.)

$P(5) =$ 10 $\qquad$ $P(P(5)) =$ $2 + 5e^{-5}$

b. [4 points] Below, you are given a table with some data about two functions: $f(t)$ and $h(t)$. You are also given information about some transformations and combinations of these functions. Fill in the missing entries in the table. You may assume $f(t)$ and $h(t)$ are invertible functions. *No work or explanation is required.*

t	0	1	2	3
$f(t)$	2	4	5	9
$h(t)$	3	8	1	7
$f(h(t))$	9	6	4	11
$f^{-1}(t)$	12	11	0	10
$f(t+3)$	9	7	8	12

c. [4 points] Suppose $g(x)$ is a power function such that $g(1) = 3$ and $g(5) = 6$. Find a formula for $g(x)$ in terms of x. *Give your answer in exact form.*

Solution: Since $g(x)$ is a power function, it can be written in the form $g(x) = kx^p$ for some constants k and p.
Using the given data, we have $g(1) = 3$, so $3 = k(1^p)$ and thus $3 = k$. To find p, we use the fact that $g(5) = 6$ and find $6 = 3(5^p)$, so $2 = 5^p$. Taking the natural log of both sides of this equation, we see that $\ln 2 = p \ln 5$ so $\ln 2/\ln 5 = p$. Thus a formula for $g(x)$ is $g(x) = 3x^{\ln 2/\ln 5}$.

$g(x) =$ $3x^{\ln 2/\ln 5}$

6. [12 points] *Remember to show your work carefully. All numbers appearing in your answers should either be in* exact form *or be accurate to at least 3 decimal places.*

Authorities in Volterra, Italy noticed an increase in the sales of extra-strength dental floss at supermarkets in the city during the early part of the year 2010. Let $D = p(t)$ denote the quantity of extra-strength dental floss, in meters, sold in Volterra on day t of 2010 (where $t = 1$ represents January 1). We are told that $p(t)$ is an exponential function.

a. [5 points] 400 meters of extra-strength dental floss were sold in Volterra on January 7, and 600 meters of extra-strength dental floss were sold on January 23. Find a formula for $p(t)$ in terms of t.

Solution: Since $p(t)$ is exponential, there are constants a and b so that $p(t) = ab^t$. The information provided tells us that $p(7) = 400$ and $p(23) = 600$, so $400 = ab^7$ and $600 = ab^{23}$. Dividing, we see that $\frac{600}{400} = \frac{ab^{23}}{ab^7}$ so $1.5 = b^{16}$. Thus $b = \sqrt[16]{1.5} \approx 1.026$. To find a, we use the value of b we just found and the equation $400 = ab^7$ to see that $400 = a(1.5)^{7/16}$ so $a = 400(1.5^{-7/16}) \approx 334.981$. Hence a formula for $p(t)$ is $p(t) = 400(1.5^{-7/16})(\sqrt[16]{1.5})^t = 400(1.5^{(t-7)/16}) \approx 334.981(1.026)^t$.

$p(t) =$ $400(1.5^{-7/16})(\sqrt[16]{1.5})^t = 400(1.5^{(t-7)/16}) \approx 334.981(1.026)^t$

b. [3 points] How long does it take for the quantity of extra-strength dental floss sold each day to double?

Solution: With a and b as in part (a), we want to find t so that $p(t) = 2p(0)$, i.e. so that $ab^t = 2a$ or $b^t = 2$. Using the natural logarithm, we see that $t \ln b = \ln 2$ so $t = \ln 2/\ln b = \ln 2/\ln(\sqrt[16]{1.5}) = 16 \ln 2/\ln 1.5 \approx 27.352$. Hence the quantity of extra-strength dental floss sold each day doubles in $16 \ln 2/\ln 1.5$ (just over 27) days.

Answer: $\frac{16 \ln 2}{\ln 1.5}$ **days**

c. [4 points] Sales of flea shampoo have also been increasing. If $F = q(t)$ is the quantity of flea shampoo, in grams, sold in Volterra on day t of 2010, then $q(t) = \ln(t+1) + 65$. Find a formula for $q^{-1}(F)$ in terms of F.

Solution: We solve for t in the formula $F = \ln(t+1) + 65$ and find

$$\begin{aligned} F &= \ln(t+1) + 65 \\ F - 65 &= \ln(t+1) \\ e^{F-65} &= t+1 \\ e^{F-65} - 1 &= t \end{aligned}$$

So $q^{-1}(F) = e^{F-65} - 1$.

$q^{-1}(F) =$ $e^{F-65} - 1$

7. [8 points] Consider the three functions described below.

- The local animal shelter has a number of dogs available that people can adopt for free. The weight of a dog at the animal shelter is a function of its length. Let $f(L)$ be the weight, in pounds, of a dog at the animal shelter that is L inches long.
- There is also a dog washing service. The amount they charge to wash a dog is a function of the dog's weight. Let $g(W)$ be the price, in dollars, they charge to wash a dog that weighs W pounds.
- The amount of food a dog eats is a function of the dog's weight. Let $h(W)$ be the cost, in dollars, of a month's supply of food for a dog that weighs W pounds.

Assume that f, g, and h are invertible functions. Fill in each blank below with an appropriate expression. The expression may involve one or more of the functions defined above.

Example: If you have a dog that weighs 29 pounds, it will cost ___$h(29)$___ dollars to buy a month's supply of food for your dog.

a. [2 points] You are considering adopting a dog that is 34 inches long. That dog weighs

___$f(34)$___ pounds.

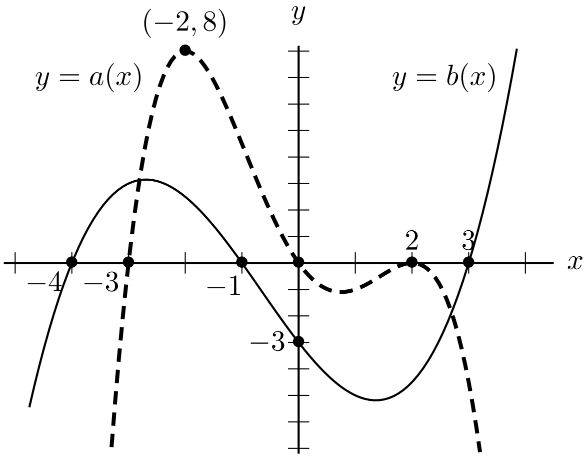
b. [2 points] You have a dog that weighs 25 pounds. If you get your dog washed, and then buy a month's supply of food for it, you will spend a total of ___$g(25) + h(25)$___ dollars.

c. [2 points] For $30, you can buy a month's supply of food for a dog that weighs

___$h^{-1}(30)$___ pounds.

d. [2 points] If you adopt a dog that is 18 inches long and want to get it washed, it will cost you ___$g(f(18))$___ dollars.

8. [12 points] The graphs of two polynomials $a(x)$ (dashed line) and $b(x)$ (solid line) are shown below. Assume all the key features of the graphs are shown. Note: No work or explanation is required for parts (a)–(c). However, partial credit *may* be awarded for work shown.



a. [2 points] Evaluate $b(a(2))$.

Solution: From the graph, we see that $b(a(2)) = b(0) = -3$.

Answer: $b(a(2)) = -3$

b. [4 points] Find the zero(s) and vertical asymptote(s) of the function $\dfrac{a(x)}{b(x)}$.

Solution: Note that $a(x)$ and $b(x)$ have no common zeros.
The zeros of $\dfrac{a(x)}{b(x)}$ are the zeros of the numerator $a(x)$, i.e. $x = -3, 0$, and 2.
The vertical asymptotes of $\dfrac{a(x)}{b(x)}$ are given by the zeros of the denominator $b(x)$, so the vertical asymptotes are $x = -4$, $x = -1$, and $x = 3$.

Zero(s): $x = -3, 0, 2$ **Vertical asymptote(s):** $x = -4$, $x = -1$, $x = 3$

c. [2 points] Estimate the horizontal intercept(s) of the function $a(x) - b(x)$.

Solution: The horizontal intercepts of the function $a(x) - b(x)$ are given by the two points of intersection of the graphs of $y = a(x)$ and $y = b(x)$. One of these occurs at $x \approx -2.7$ and the other at $x \approx 2.7$.

Horizontal intercept(s): $x \approx -2.7, 2.7$

d. [4 points] Find a possible formula for the polynomial $a(x)$. You do not need to simplify your answer. *Show your work.*

Solution: Note that the long-run behavior of the graph shows that the degree of $a(x)$ is even, and since the graph "turns" three times, the degree must be at least four.
The polynomial $a(x)$ has zeros at $x = -3$ and $x = 0$ and a zero of even multiplicity at $x = 2$. A possible formula for $a(x)$ is then $a(x) = k(x+3)(x)(x-2)^2$. Since the point $(-2, 8)$ is on the graph of $y = a(x)$, we see that $a(-2) = 8$, so $8 = k(-2+3)(-2)(-2-2)^2$. Thus $8 = k(-32)$ so $k = -1/4$. Hence a possible formula for $a(x)$ is $a(x) = -\frac{1}{4}x(x+3)(x-2)^2$.

$a(x) = -\dfrac{1}{4}x(x+3)(x-2)^2$

9. [7 points] In the United States, the number of werewolves, W, living in a given state is a function $W = g(V)$ of the number of vampires, V, that live in that state. The formula for $g(V)$ is $g(V) = kV^{2/3}$, where k is a positive constant. The constant k does not depend on the state.

a. [4 points] In Pennsylvania, there are 1728 vampires and 720 werewolves. In Indiana, there are 512 vampires. How many werewolves live in Indiana?

Solution: The data from Pennsylvania show that $g(1728) = 720$. Therefore, $1720 = k(1728)^{2/3}$, so $720 = k144$ and $k = 1720/144 = 5$. Thus, the formula for $g(V)$ is $g(V) = 5V^{2/3}$. Since there are 512 vampires in Indiana, we compute that the number of werewolves in Indiana is $g(512) = 5(512^{2/3}) = 320$.

Answer: 320 **werewolves**

b. [3 points] There are 50% more vampires in Ohio than there are in Michigan. How much larger is the werewolf population of Ohio than that of Michigan?
Your answer should be accurate to at least 0.01%.

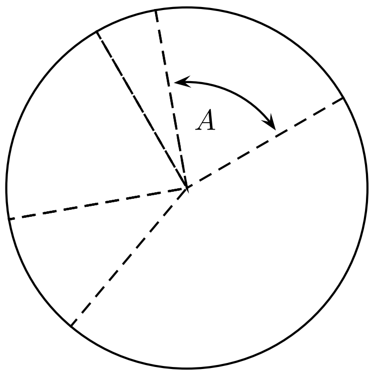
Solution: Let V_M be the number of vampires in Michigan. Then the number of vampires in Ohio is $1.5V_M$. So the number of werewolves in Michigan is $kV_M^{2/3}$ and the number of werewolves in Ohio is $k(1.5V_M)^{2/3}$. Hence the ratio of the number of werewolves in Ohio to the number of werewolves in Michigan is

$$\frac{k(1.5V_M)^{2/3}}{kV_M^{2/3}} = \frac{(1.5)^{2/3}V_M^{2/3}}{V_M^{2/3}} = 1.5^{2/3} \approx 1.31037.$$

Hence, there are approximately 31.04% more werewolves in Ohio than there are in Michigan.

Answer: The werewolf population of Ohio is 31.04% percent larger than the werewolf population of Michigan.

10. [9 points] There is a pumpkin pie in the shape of a circle of radius 12 centimeters. The pie is sliced by making cuts along radii, as pictured below. (Slices are NOT necessarily the same size.) *Show your work. All answers should be in exact form or be accurate to at least 3 decimal places.*



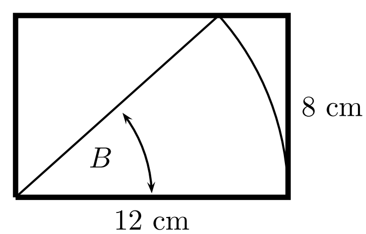
Note: Figure (and slices) not drawn to scale.

a. [2 points] The pie is surrounded by a thin crust. The first slice of pie you take has angle measuring A radians. (See picture.) If the length of crust on your slice is 9 centimeters, compute the value of A.

Solution: Using the arclength formula, we have $9 = 12A$, so $A = 9/12 = 0.75$.

$A =$ 0.75

b. [3 points] You have a plate in the shape of a rectangle of length 12 cm and width 8 cm. Your second slice of pie has angle measuring B, in radians. Find the maximum value of B so that your slice of pie will fit on the plate (as shown in the picture below).

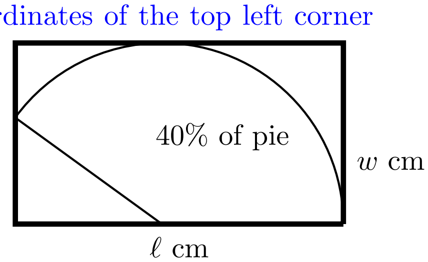


Solution: If we think of the center of the pie as the origin $(0,0)$, then the coordinates of the top right corner of the piece of pie are $(12\cos B, 12\sin B)$. So, the largest possible value of B satisfies $12\sin B = 8$. That is, $\sin B = 8/12$ and $0 \le B \le \pi/2$, so $B = \arcsin(8/12) \approx 0.7297$.

Answer: $\arcsin(2/3) \approx 0.7297$

c. [4 points] At the end of the evening, after helping yourself to several slices, 40% of the pie remains. The pie will be placed in a rectangular tupperware container (as shown in the picture below) to be refrigerated. Find the length, ℓ, and width, w, both measured in centimeters, of the smallest tupperware container into which the remaining pie will fit.

Solution: First note that $w = 12$ since at least 25% of the pie remains. Now, the angle measurement of the remaining 40% of the pie is 40% of 2π which is $0.4(2\pi) = 0.8\pi$. If we let the center of the pie be the origin $(0,0)$, then the coordinates of the top left corner of the remaining pie shown are $(12\cos(0.8\pi), 12\sin(0.8\pi))$. Hence the coordinates of the lower corners of the rectangular container are $(12\cos(0.8\pi), 0)$ and $(12, 0)$. Thus $\ell = 12 - 12\cos(0.8\pi) \approx 21.708$.



$w =$ 12

$\ell =$ $12 - 12\cos(0.8\pi) \approx 21.708$