Math 105 — First Midterm

October 8, 2012

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 8 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	10	
3	4	
4	15	
5	13	
6	5	
7	10	
8	15	
9	14	
Total	100	

1. [14 points] A student named U.M. Student needs to choose a data plan for her new smartphone. She has three plans to choose from.

- The **Maize** Plan costs \$30 per month plus an additional \$10 per gigabyte of data used.
- The **Blue** Plan costs \$15 per month plus an additional \$20 per gigabyte of data used.
- The **Wolverine** Plan costs \$100 per month including unlimited data usage.

a. [4 points] Let $M(g)$, $B(g)$, and $W(g)$ be the total cost, in dollars, for a month in which she uses g gigabytes of data under the Maize, Blue, and Wolverine Plans, respectively. Find a formula for each of these functions.

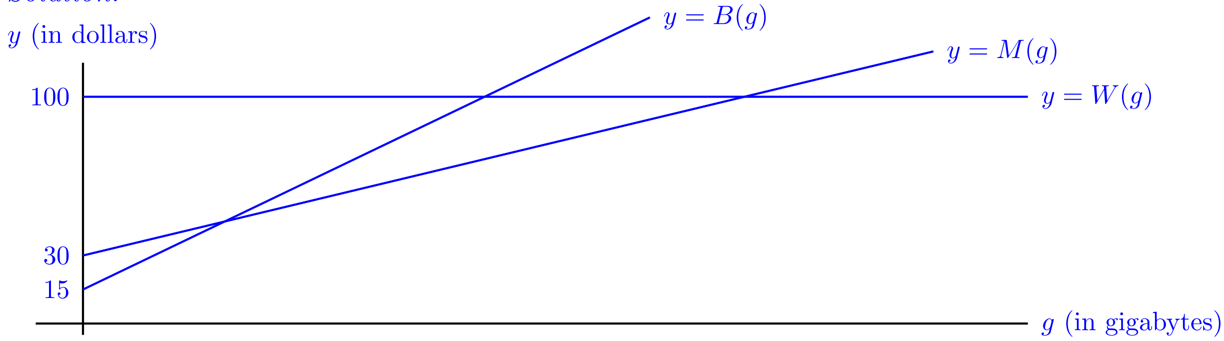
$M(g) =$ $30 + 10g$

$B(g) =$ $15 + 20g$

$W(g) =$ 100

b. [4 points] Sketch the graphs of each of these three functions on the axes below. Be sure to label the axes appropriately, including the values of any intercepts, and clearly indicate which graph is which.

Solution:



c. [2 points] If U.M. Student expects to use 1 gigabyte of data per month, which plan would be cheapest? (Justify your answer.)

Solution: $M(1) = 30 + 10(1) = 40$, $B(1) = 15 + 20(1) = 35$, and $W(1) = 100$, so the Blue Plan would be cheapest. (The price of one month of service using 1 gigabyte of data under the Maize, Blue, and Wolverine Plans are \$40, \$35, and \$100, respectively.)

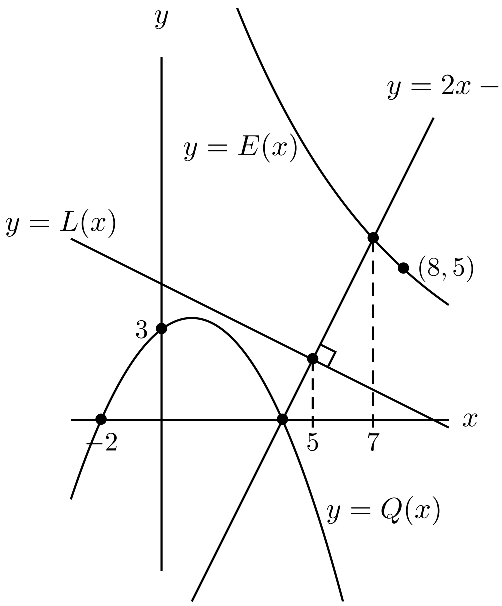
d. [4 points] Under what circumstances is the Maize Plan the cheapest? (In other words, for exactly what quantities of monthly data usage is the Maize Plan the cheapest of the three options?) *Show your work and/or explain your reasoning clearly.*

Solution: From our graph above, we see that the Maize Plan is the cheapest for values of g between the point of intersection of $y = M(g)$ and $y = B(g)$ and the point of intersection of $y = M(g)$ and $y = W(g)$. Solving for these points of intersection, we have

$$\begin{aligned} M(g) &= B(g) \\ 30 + 10g &= 15 + 20g \\ -10g &= -15 \\ g &= 1.5 \end{aligned} \qquad\qquad \begin{aligned} M(g) &= W(g) \\ 30 + 10g &= 100 \\ 10g &= 70 \\ g &= 7 \end{aligned}$$

So, the Maize Plan is cheapest for $1.5 < g < 7$, i.e. it is cheapest if U.M. will use between 1.5 and 7 gigabytes of data per month.

2. [10 points] The graph of $y = 2x - 8$ and of three functions L, Q, and E are shown below. Note that L is linear, Q is quadratic, and E is exponential.
Use the information shown in the graph to find formulas for $L(x)$, $Q(x)$, and $E(x)$.
Graphs may not be drawn to scale, so be careful! Use only the information that is labeled in the graph. Show your work clearly and leave all numbers in EXACT FORM.



Remember: Show your work clearly in the space on this page, and leave all numbers in EXACT FORM. *Write your final answers in the answer blanks below.*

$L(x) =$ $4.5 - \frac{1}{2}x$

$Q(x) =$ $-\frac{3}{8}(x+2)(x-4)\left(= -\frac{3}{8}x^2 + \frac{3}{4}x + 3\right)$

$E(x) =$ $\frac{6^8}{5^7}\left(\frac{5}{6}\right)^x$

Solution:
$\boxed{L(x)}$ First, note that the graph of $y = L(x)$ is perpendicular to the graph of $y = 2x - 8$, so the slope of the linear function L is $-1/2$. The y–coordinate of their point of intersection is $2(5) - 8 = 2$, so $(5, 2)$ is a point on the graph of $y = L(x)$. Using point-slope form, we have $L(x) - 2 = -\frac{1}{2}(x - 5)$ so $L(x) = 2 - \frac{1}{2}(x - 5) = 4.5 - \frac{1}{2}x$.

$\boxed{Q(x)}$ Solving $0 = 2x - 8$, we find that the x-intercept of $y = 2x - 8$ is 4. So the two zeros of Q are -2 and 4. Hence a formula for $Q(x)$ is $Q(x) = a(x + 2)(x - 4)$ for some constant a. Since the y-intercept is 3 we see that $3 = a(0 + 2)(0 - 4)$, so $3 = -8a$ and $a = -3/8$. Thus $Q(x) = -\frac{3}{8}(x + 2)(x - 4)$.

$\boxed{E(x)}$ The point of intersection of $y = E(x)$ and $y = 2x - 8$ is $(7, 6)$ (since $2(7) - 8 = 6$), so two points on the graph of $y = E(x)$ are $(7, 6)$ and $(8, 5)$. Since E is exponential and $8 - 7 = 1$, the growth/decay factor of E is $E(8)/E(7) = 5/6$. A formula for $E(x)$ is then $E(x) = c(5/6)^x$ for some constant c. Using the point $(7, 6)$ we find that $6 = c(5/6)^7$ so $c = 6(6/5)^7$. Thus $E(x) = 6(6/5)^7(5/6)^x$.

3. [4 points] Find the average rate of change of the function $g(t) = 2t^2 - 3t + 4$ between $t = -1$ and $t = -1 + h$. *For full credit, simplify your answer as much as possible.*

Solution: The average rate of change is

$$\frac{g(-1+h) - g(-1)}{(-1+h) - (-1)} = \frac{\left(2(-1+h)^2 - 3(-1+h) + 4\right) - \left(2(-1)^2 - 3(-1) + 4\right)}{(-1+h) + 1}$$

$$= \frac{2(1 - 2h + h^2) - 3(-1+h) + 4 - (2 + 3 + 4)}{h}$$

$$= \frac{2h^2 - 7h}{h} = \frac{h(2h - 7)}{h} = \boxed{\text{2h-7}}.$$

4. [15 points]
A load of bricks is being lifted by a crane at a constant speed of 5.6 m/s. A brick falls off the stack. The fallen brick's height, in meters above the ground, t seconds after falling off the stack is given by $h(t) = 15.4 + 5.6t - 4.9t^2$.

Throughout this problem, remember to include units and show your work and/or explain your reasoning clearly. (Recall Instruction #7 from the front page.) All answers should be given either in exact form or to at least two decimal places.

a. [2 points] How high above the ground was the brick when it fell off the stack?

Solution: The brick fell off the stack at time $t = 0$ and $h(0) = 15.4$, so the brick was $\boxed{\text{15.4 meters}}$ above the ground when it fell off the stack.

b. [3 points] How long does it take for the brick to hit the ground?

Solution: If the brick hits the ground at time g, then $h(g) = 0$, so we solve for g in the equation $15.4 + 5.6g - 4.9g^2 = 0$. By the quadratic formula, solutions to this equation are given by $g = \dfrac{-5.6 \pm \sqrt{5.6^2 - 4(-4.9)(15.4)}}{2(-4.9)} = \dfrac{-5.6 \pm \sqrt{333.2}}{-9.8} = \dfrac{5.6 \pm \sqrt{333.2}}{9.8}$. These two solutions are approximately equal to -1.291 and 2.434. Only the postive solution makes sense in the context of this problem. So, the brick hits the ground $\boxed{\text{approximately 2.434 seconds}}$ after it falls from the stack.
(Note: Alternatively, we could use a graphing calculator to find the positive zero of the function $h(t)$.)

c. [3 points] When does the brick reach its highest point?
How high above the ground is the brick at that time?

Solution: Since the graph of h opens downward, h reaches its maximum at its vertex, so the brick reaches its highest point at the t–coordinate of its vertex, which is $\frac{-5.6}{2(-4.9)} = \frac{4}{7}$. (We can find this by beginning the process of completing the square, i.e. $h(t) = -4.9(t^2 + \frac{5.6}{4.9}t) + 15.4$ so the t-coordinate of the vertex is at $t = \frac{1}{2}(\frac{5.6}{4.9})$, or by using the "maximum" feature of the graphing calculator.) At $t = 4/7$, the height of the brick is $h(4/7) = 17$ meters. So, $\boxed{4/7 \text{ seconds}}$ after falling from the stack, the brick reaches its highest point, which is $\boxed{17 \text{ meters}}$ above the ground

d. [3 points] Find the domain and range of the function h in the context of this problem.

Solution: In the context of this problem, the domain is approximately $[0, 2.434]$ (based on part (b)) and the range is $[0, 17]$ (based on part (c)).

Domain: $[0, 2.434]$ **Range:** $[0, 17]$

e. [4 points] The supervisor of the construction site sees the brick fall as it passes in front of his office window, which is at a height of 3 meters above the ground. How much time passes between when the supervisor sees the brick and when the brick hits the ground?

Solution: To solve $h(t) = 3$, we can use the quadratic equation (to solve $-4.9t^2 + 5.6t + 12.4 = 0$) or the "intersect" feature of the graphing calculator to find $t \approx 2.2617$, so the supervisor sees the brick approximately 2.2617 seconds after it falls, from the stack. This is about $2.4341 - 2.2617 = 0.1724$ seconds before the brick hits the ground. Hence the supervisor sees the brick about $\boxed{0.172 \text{ seconds}}$ before the brick hits the ground.

5. [13 points] In 1940, there were 6.1 million farms in the United States, and this number decreased by a total of 60% during the next 40 years.

a. [2 points] Based on the data above, how many farms were there in the US in 1980?

Solution: In 1980, there were 40% as many farms as there were in 1940, so there were $0.4(6.1) = \boxed{2.44 \text{ million farms}}$.

b. [5 points] Suppose that the number of farms decreased *at a constant rate* from 1940–1980. Find a formula for $F(t)$, the number of millions of farms in the US this model predicts there were t years after 1940 .

Solution: Since the rate of change is constant, F is linear. The constant average rate of change (slope) of F is $\dfrac{F(40) - F(0)}{40 - 0} = \dfrac{2.44 - 6.1}{40} = -0.0915$ million farms per year. Since $F(0) = 6.1$, we use slope-intercept form to see that $\boxed{F(t) = 6.1 - 0.0915t}$.

According to this model, in what year were there (or will there be) a total of 4 million farms in the US?

Solution: We solve the equation $F(t) = 4$ and find

$$\begin{aligned} F(t) &= 4 \\ 6.1 - 0.0915t &= 4 \\ -0.0915t &= -2.1 \\ t &\approx 22.95 \end{aligned}$$

So according to this model, there were 4 million farms in the US in about 1963.

c. [6 points] Now, suppose instead that the number of farms decreased *at a constant percent rate* from 1940–1980. Under this new assumption, by what percent did the number of farms in the US decrease each year between 1940 and 1980?

Solution: Under this assumption, the number of farms is an exponential function of time. Let b be the annual decay factor. Then the number of farms in 1980 was $6.1(b^{40})$, so $2.44 = 6.1(b^{40})$. Thus $b^{40} = 0.4$ so $b = 0.4^{1/40} \approx 0.97735$. Hence the number of farms in the US decreased by about 2.23% each year between 1940 and 1980.

Find a formula for $P(t)$, the number of millions of farms in the US this model predicts there were t years after 1940 .

Solution: This is the formula we were working with above. In particular, this is an exponential function with initial value 6.1. We found the annual decay factor b above, so we have $\boxed{P(t) = 6.1(0.4)^{t/40} \approx 6.1(0.9774)^t}$.

6. [5 points] Let $f(x) = -4x^2 + 12kx - 17$. Use the method of completing the square to rewrite this function in vertex form and then give the coordinates of the vertex.
Show your work step-by-step. Note: Your answers may involve the constant k.

Solution:

$$\begin{aligned} f(x) &= -4x^2 + 12kx - 17 \\ &= -4(x^2 - 3kx) - 17 \\ &= -4\left[x^2 - 3kx + \left(\frac{-3k}{2}\right)^2 - \left(\frac{-3k}{2}\right)^2\right] - 17 \\ &= -4\left[\left(x - \frac{3k}{2}\right)^2 - \frac{9k^2}{4}\right] - 17 = -4\left(x - \frac{3k}{2}\right)^2 + 9k^2 - 17 \end{aligned}$$

Vertex form: $f(x) = -4\left(x - \frac{3k}{2}\right)^2 + (9k^2 - 17)$

Vertex: $(3k/2, 9k^2 - 17)$

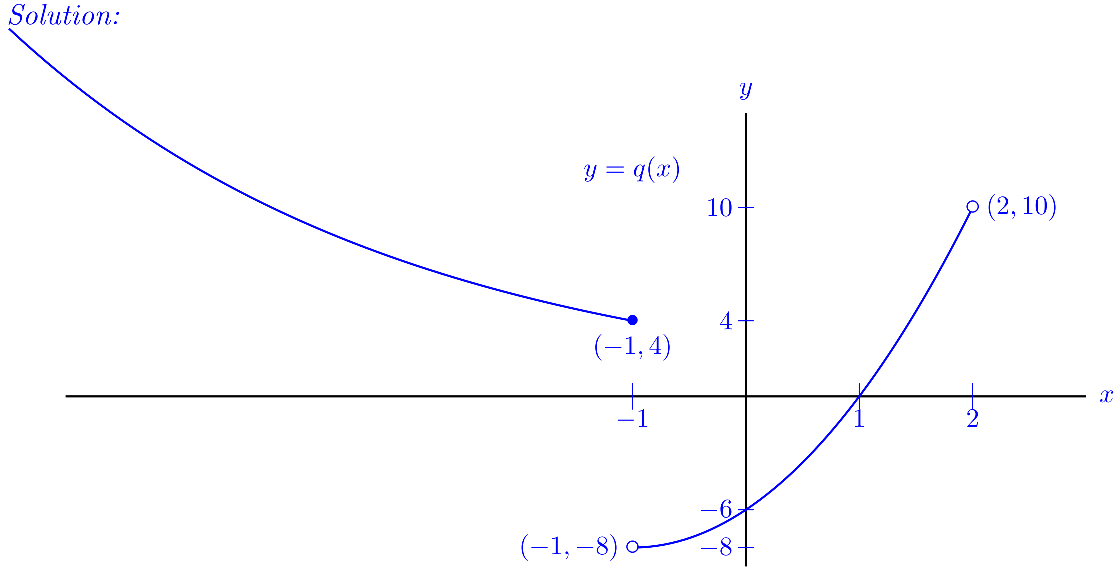
7. [10 points] Consider the function q defined by $q(x) = \begin{cases} 3(0.75)^x & \text{if } \quad x \leq -1 \\ 2(x+1)^2 - 8 & \text{if } \ -1 < x < 2 \end{cases}$

a. [2 points] Evaluate $q(q(0))$.

Solution: $q(q(0)) = q(2(0+1)^2 - 8) = q(2-8) = q(-6) = \boxed{3(0.75)^{-6}}$.

b. [4 points] Sketch a graph of $y = q(x)$. Carefully label your axes and important points on your graph (including intercepts).

Solution:



c. [4 points] Find the domain and range of q. (Use either interval notation or inequalities.)

Solution: Based on the given formula, we see that the domain of q is the interval $(-\infty, 2)$ and using the graph from part (b), we conclude that the range of q is the interval $(-8, \infty)$.

Domain: $(-\infty, 2)$ **Range:** $(-8, \infty)$

8. [15 points] The cost of computer memory has changed dramatically over time. Let $C(t)$ be the cost, in dollars per gigabyte, of computer memory t years after 1956. Some estimated data for C is provided in the table below.[1]

t	0	33	38	44	48	55
$C(t)$	10,000,000	36,000	1000	20	1	0.035

a. [3 points] Find and interpret, in the context of this problem, the average rate of change of $C(t)$ for $33 \le t \le 38$. (Use a complete sentence and include units.)

Solution: The average rate of change is $\frac{C(38)-C(33)}{38-33} = \frac{1000-36000}{5} = -7000$ dollars per gigabyte per year. So, between 1989 and 1994, the cost of computer memory decreased at an average rate of \$7000 per gigabyte per year.

Note: We collect the successive average rates of change of C for reference in parts (b)–(d) below.

interval	$0 \le t \le 33$	$33 \le t \le 38$	$38 \le t \le 44$	$44 \le t \le 48$	$48 \le t \le 55$
Δt (in years)	33	5	6	4	7
$\Delta C(t)$ (in \$/GB)	-9964000	-35000	-980	-19	-0.065
Avg rate of change (in \$/GB per yr)	≈ -301939.39	-7000	≈ -163.3	-4.75	≈ -0.00929

b. [4 points] Based on the data provided in the table above, could the function $C(t)$ be linear, exponential, or neither linear nor exponential? (*Circle one.*)

Linear Exponential **[Neither linear nor exponential]**

Justify your answer numerically (i.e. show your work and explain your reasoning).

Solution: The average rate of change is *not* constant (as can be seen in the table above), so the function is *not* linear.
Note that $C(44)/C(33) \approx 0.00056$ whereas $C(55)/C(44) = 0.00175$. Since the two time intervals $33 \le t \le 44$ and $44 \le t \le 55$ are both the same length (11 years), these ratios would be the same if $C(t)$ were exponential. Therefore $C(t)$ is *not* exponential. (Note that alternatively, we could have computed the annual decay factor over each time interval in the table to see that this factor is not constant.)

c. [2 points] Based on the data provided in the table above, is the function $C(t)$ increasing, decreasing, or neither increasing nor decreasing on the entire interval from $t = 0$ to $t = 55$? (*Circle one.*)

Increasing **[Decreasing]** Neither increasing nor decreasing

Solution: The average rate of change over every time interval shown in the table is negative, so $C(t)$ appears to be decreasing over the entire interval from $t = 0$ to $t = 55$.

d. [2 points] Based on the data provided in the table above, is the function $C(t)$ concave up, concave down, or neither concave up nor concave down on the entire interval from $t = 0$ to $t = 55$? (*Circle one.*)

[Concave Up] Concave Down Neither concave up nor concave down

Solution: The average rate of change of $C(t)$ over successive time intervals is increasing (becoming "less negative"), so $C(t)$ appears to be concave up.

e. [4 points] Estimate $C^{-1}(46)$. Then interpret its meaning in the context of this problem. (Use a complete sentence and include units.)

Solution: $C^{-1}(46)$ is between 38 and 44, most likely closer to 44 (since 46 is much closer to 20 than to 1000). So, we estimate that $C^{-1}(46) \approx 43$.
This means that the cost of memory was 46 dollars per gigabyte in approximately 1999.

[1]Source: http://en.wikipedia.org/wiki/Memory_storage_density

9. [14 points] A fashion designer has a budget of \$300 for fabric for a fabulous garment. The designer is going to use a combination of denim fabric which costs \$8 per yard and jersey fabric which costs \$12 per yard. (Assume that the fabric store will sell any length of these fabrics, i.e. partial yards are okay.)

Assume that the designer spends the entire budget of \$300 on these two fabrics. Let D be the number of yards of denim and J be the number of yards of jersey that the designer purchases.

a. [2 points] In one complete sentence, explain why J is a function of D.

Solution: Each value of the input D determines exactly one value of the output J because once the designer decides on D, J is completely determined by the amount of money from the budget that is left over.

Let $f(D)$ be the number of yards of jersey that the designer buys if the designer buys D yards of denim, so $J = f(D)$.

b. [3 points] Evaluate $f(5)$ and interpret it in the context of this problem. (Use a complete sentence and include units.)

Solution: $f(5)$ is the number of yards of jersey that the designer buys if he/she buys 5 yards of denim.

5 yards of denim costs a total of \$40, leaving \$260 for jersey. Each yard of jersey costs \$12, so the designer will buy $260/12 = 21\ 2/3$ yards of jersey. Hence $\boxed{f(5) = 21\ 2/3}$.

Interpretation: If the designer buys 5 yards of denim, then he/she buys 21 2/3 yards of jersey.

c. [3 points] Find a formula for $f(D)$.

Solution: If the designer buys D yards of denim and J yards of jersey then he/she spends \$$8D$ on denim and \$$12J$ on jersey. Because the designer spends the entire budget on denim and jersey, we have $8D+12J = 300$ so solving for J we find $J = \frac{300-8D}{12} = 25-\frac{2}{3}D$. Thus $\boxed{f(D) = 25 - \frac{2}{3}D}$.

d. [3 points] Find and interpret, in the context of this problem, the D–intercept of the graph of $J = f(D)$. (Use a complete sentence and include units.)

Solution: The D-intercept is the value of D when $J = 0$, which is the solution to $8D + 12(0) = 300$ or $\boxed{D = 37.5}$.

So, the designer buys 37.5 yards of denim if he/she buys no jersey.

e. [3 points] Give a practical interpretation of $f^{-1}(k)$ in the context of this problem. (Use a complete sentence and include units. You do not need to find a formula.)

Solution: $f^{-1}(k)$ is the number of yards of denim the designer buys if he/she buys k yards of jersey. (Another phrasing: If the designer buys k yards of jersey, then he/she buys $f^{-1}(k)$ yards of denim.)