Math 105 — Second Midterm

November 12, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	6	
3	5	
4	6	
5	10	
6	14	
7	12	
8	10	
9	11	
10	8	
11	10	
Total	100	

1. [8 points] For each of the statements below, circle "**True**" if the statement is *definitely* true. Otherwise, circle "**False**". You do not need to show any work for this problem.

a. [2 points] The function $g(x) = 3^x + \left(\frac{1}{3}\right)^x$ is an even function.

True False

b. [2 points] The graph of $y = \ln(10x)$ can be obtained from the graph of $y = \ln(x)$ by a vertical shift.

True False

c. [2 points] The line $y = 3$ is a horizontal asymptote of the function $f(x) = e^{10000x} + 3$.

True False

d. [2 points] The function $h(x) = 5\cos(3x)$ is an odd function.

True **False**

2. [6 points] The graph of the function $g(x)$ contains the point $(-6, 4)$. For each of the functions below, find the coordinates of one point that must be on the graph of the function. Write the coordinates of the point in the form (x, y) on the provided answer blank.
You do not have to show work for this problem.

a. [2 points]
If $h(x) = 0.25g(-0.5x)$, then the graph of $h(x)$ must contain the point $(12, 1)$.

Solution: The graph of $h(x) = 0.25g(-0.5x)$ can be obtained from the graph of $g(x)$ as follows:
- First, compress vertically by a factor of 0.25, taking the point $(-6, 4)$ to the point $(-6, 1)$.
- Next, stretch horizontally by a factor of 2 $(= \frac{1}{0.5})$, taking the point $(-6, 1)$ to the point $(-12, 1)$).
- Finally, reflect across the vertical axis, taking the point $(-12, 1)$ to the point $(12, 1)$.

Check: $h(12) = 0.25g(-0.5(12)) = 0.25g(-6) = 0.25(4) = 1$, so $(12, 1)$ is indeed on the graph of $h(x)$.

b. [2 points]
If $n(x) = g(x+3)-4$, then the graph of $n(x)$ must contain the point $(-9, 0)$.

Solution: The graph of $n(x) = g(x + 3) - 4$ can be obtained from the graph of $g(x)$ as follows:
- Shift the graph *down* 4 units, taking the point $(-6, 4)$ to the point $(-6, 0)$.
- Shift the resulting graph 3 units *to the left*, taking the point $(-6, 0)$ to the point $(-9, 0)$.

Check: $n(-9) = g(-9 + 3) - 4 = g(-6) - 4 = 4 - 4 = 0$, so $(-9, 0)$ is indeed on the graph of $n(x)$.

c. [2 points]
If $p(x) = -3g(2x-4)$, then the graph of $p(x)$ must contain the point $(-1, -12)$.

Solution: By factoring, we rewrite $p(x) = -3g(2x - 4)$ as $p(x) = -3g(2(x - 2))$. The graph of $p(x) = -3g(2(x - 2))$ can be obtained from the graph of $g(x)$ as follows:
- First, stretch vertically by a factor of 3, taking the point $(-6, 4)$ to the point $(-6, 12)$.
- Next, reflect across the horizontal axis, taking the point $(-6, 12)$ to the point $(-6, -12)$.
- Then, compress horizontally by a factor of $1/2$, taking $(-6, -12)$ to the point $(-3, -12)$.
- Finally, shift the resulting graph 2 units *to the right*, taking $(-3, -12)$ to the point $(-1, -12)$.

Check: $p(-1) = -3g(2(-1) - 4) = -3g(-6) = -3(4) = -12$, so $(-1, -12)$ is indeed on the graph of $p(x)$.

3. [5 points] A colony of bacteria triples in size every 6 days. What is the doubling time of this colony? (Show your work step-by-step, give your final answer in **exact form**, and *include units.*)

Solution: The colony is growing exponentially, so if its initial size is a, then its size after t days is ab^t for some constant b. Since the colony triples in size every 6 days, its population when $t = 6$ is $3a$, so $3a = ab^6$. Then $3 = b^6$ so $b = (3)^{1/6}$ and the colony size after t days is $a\left(3^{1/6}\right)^t = a(3^{t/6})$.

Let d be the doubling time of the colony. Then $2a = a(3^{d/6})$ so $2 = 3^{d/6}$.
Taking the natural logarithm of both sides of this equation and solving for d we find

$$\begin{aligned} 2a &= a(3^{d/6}) \\ 2 &= 3^{d/6} \\ \ln(2) &= \ln\left(3^{d/6}\right) \\ \ln(2) &= \frac{d}{6}\ln(3) \\ \frac{6\ln(2)}{\ln(3)} &= d \end{aligned}$$

Hence the doubling time of this colony is $\dfrac{6\ln(2)}{\ln(3)}$ days. (This is approximately 3.79 days.)

Answer: $\dfrac{6\ln(2)}{\ln(3)}$ **days**

4. [6 points] Let $G(m)$ be the mass (in grams) of the garbage in a dumpster m minutes before 8 am. For each of the functions below, find a formula by applying one or more appropriate transformations to the function G. (*In each case, your final answer should be a formula involving G.*)

a. [2 points] Let $K(m)$ be the mass (in **kilograms**) of the garbage in the dumpster m minutes before 8 am.

Answer: $K(m) =$ $0.001G(m)$.

b. [2 points] Let $L(h)$ be the mass (in kilograms) of the garbage in the dumpster h **hours** before 8 am.

Answer: $L(h) =$ $0.001G(60h)$.

c. [2 points] Let $T(h)$ be the mass (in kilograms) of the garbage in the dumpster h hours before **11 am**.

Solution: Note that $T(3) = L(0)$ (since in both cases this gives the mass in kg at 8 am). More generally, $T(h) = L(h-3) = 0.001G(60(h-3))$.

Answer: $T(h) =$ $0.001G(60(h-3))$ **or** $0.001G(60h-180)$.

5. [10 points] *For potential partial credit, be sure to show your work.*

a. [4 points] Suppose that the domain of $f(t)$ is the interval $[-10, 20)$ and the range of $f(t)$ is the interval $(-8, \infty)$. Find the domain and range of the function $h(t) = 5f(-2t) + 6$.

Solution: Note that "vertical transformations" affect only the range and "horizontal transformations" affect only the domain of the function. The graph of $h(t) = 5f(-2t)+6$ can be obtained from the graph of $f(t)$ as follows:

- Stretch vertically by a factor of 5 and then shift the resulting graph up 6 units. The range of the resulting function $(5f(t) + 6)$ is $(-34, \infty)$, and the domain is still $[-10, 20)$.
- Then reflect across the vertical axis and compress horizontally by a factor of 1/2 to get the graph of $h(t)$. The resulting range is still $(-34, \infty)$ and the domain is $(-10, 5]$.

Domain: $(-10, 5]$ **Range:** $(-34, \infty)$

b. [3 points] If a weight hanging on a string of length 6 feet swings through 11° on either side of the vertical, how long is the arc through which the weight moves from one high point to the next high point? (*Give your answer in exact form and include units.*)

Solution: Note that the total angle through which the weight swings is 22°. Using the formula for arclength, we find

$$\begin{aligned} \text{arclength} &= \text{radius} \cdot \text{angle (in radians)} \\ &= 6 \cdot \left(22^\circ \cdot \frac{\pi}{180^\circ}\right) \\ &= \frac{22\pi}{30} = \frac{11\pi}{15} \end{aligned}$$

Answer: $\frac{11\pi}{15}$ **feet**

c. [3 points] The graph of $T(x)$ can be obtained from the graph of $\tan(x)$ by

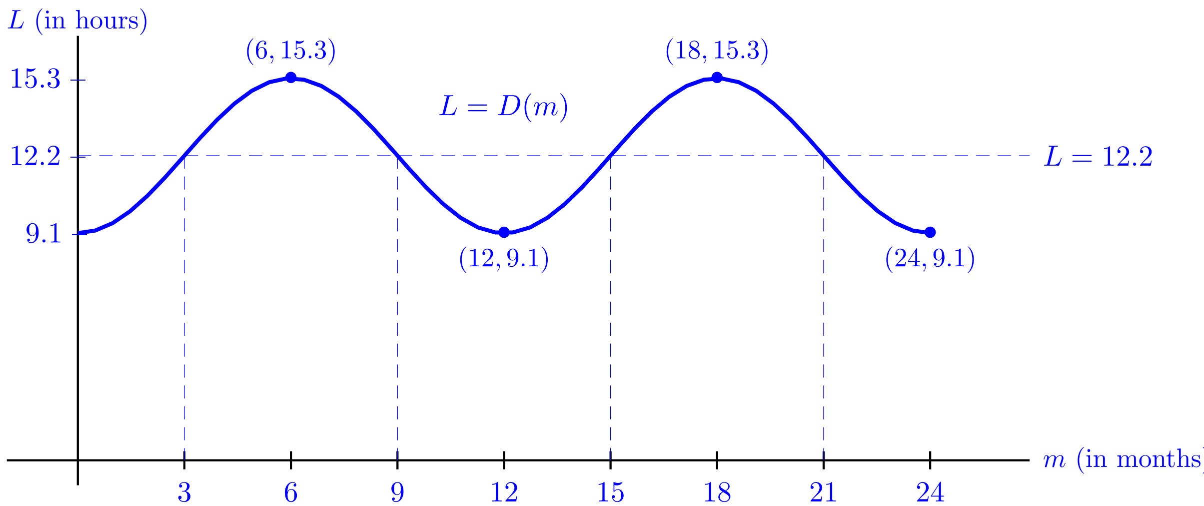
- first stretching the graph horizontally (away from the vertical axis) by a factor of 3,
- then shifting the graph to the right 5 units,
- then reflecting the graph across the horizontal axis,
- and finally shifting the graph down 2 units.

Find a formula for $T(x)$.

Answer: $T(x) = -\tan\left(\frac{1}{3}(x-5)\right) - 2$

6. [14 points] The number of hours of daylight in Ann Arbor varies from a minimum of 9.1 hours of daylight on December 21 to a maximum of 15.3 hours of daylight on June 21 (and then back down to 9.1 hours on the following December 21). Let $L = D(m)$ be the number of hours of daylight in Ann Arbor on a day that is m months after December 21, 2010. Assume that $D(m)$ is a sinusoidal function.

a. [4 points] On the axes provided below, graph *two periods* of the function $L = D(m)$ starting with $m = 0$. (*Clearly label the axes and important points on your graph. Be very careful with the* **shape** *and* **key features** *of your graph.*)



b. [4 points] Find the period, amplitude, and midline of $L = D(m)$. (*Include units* for the period and amplitude.)

Period: 12 **months**

Amplitude: 3.1 **hours**

Midline: $L = 12.2$

c. [4 points] Find a formula for $D(m)$.

Solution: Note that the graph begins at a minimum at $m = 6$ and then is increasing when it crosses the midline at $m = 3$. Using the data determined in part (b) above, two possible formulas are thus given by

$$D(m) = -3.1\cos\left(\frac{\pi}{6}m\right) + 12.2 \quad \text{and} \quad D(m) = 3.1\sin\left(\frac{\pi}{6}(m-3)\right) + 12.2.$$

Answer: $D(m) = -3.1\cos\left(\frac{\pi}{6}m\right) + 12.2$

d. [2 points] Use your formula from part (c) to estimate the number of hours of daylight in Ann Arbor on April 21. (Show your work and round your answer to the nearest 0.1 hour.)

Solution: April 21 is four months after December 21, so we need to compute $D(4)$.

$$D(4) = -3.1\cos(\frac{\pi}{6}(4)) + 12.2 = -3.1\cos(\frac{2\pi}{3}) + 12.2 = -3.1(-\frac{1}{2}) + 12.2 = 13.75.$$

Thus according to this model, there should be about 13.8 hours of daylight on April 21.

Answer: **About** 13.8 **hours of daylight**

7. [12 points] Solve each of the equations below. *Show your work step-by-step and write the solutions in* **exact form** *in the answer blanks provided.*

a. [3 points] $5(1.7)^{2y} = 2.4$

Solution: We first divide both sides of the equation by 5 and then use logarithms to find y.

$$\begin{aligned} 5(1.7)^{2y} &= 2.4 \\ (1.7)^{2y} &= \frac{2.4}{5} = 0.48 \\ \ln(1.7^{2y}) &= \ln(0.48) \\ 2y\ln(1.7) &= \ln(0.48) \\ y &= \frac{\ln(0.48)}{2\ln(1.7)} \end{aligned}$$

Answer: $y = \frac{\ln(0.48)}{2\ln(1.7)}$

b. [3 points] $3t - 1 = \log\left(2(10)^{4.6t}\right)$

Solution: Using properties of logarithms, we see that $\log\left(2(10)^{4.6t}\right) = \log(2) + \log(10^{4.6t}) = \log(2) + 4.6t$, so it remains to solve the equation $3t - 1 = \log(2) + 4.6t$. Then we find $-1.6t = \log(2) + 1$ so $t = \frac{\log(2)+1}{-1.6}$

Answer: $t = \frac{\log(2)+1}{-1.6}$

c. [3 points] $e^{\ln(w-4)} = \ln(3.2) - \ln(4)$

Solution: Applying basic properties of the natural logarithm, we see that $e^{\ln(w-4)} = w - 4$ and $\ln(3.2) - \ln(4) = \ln(\frac{3.2}{4}) = \ln(0.8)$. Thus $w - 4 = \ln(0.8)$ so $w = 4 + \ln(0.8)$. However, note that we cannot plug this value of w into the original equation (since this would involve $\ln(\ln(0.8))$, which is undefined because $\ln(0.8) < 0$). So, if there were a solution, to the equation, it would be $w = 4 + \ln(0.8)$, but there is actually no solution.

Answer: $w =$ **No solution**

d. [3 points] $\log(2p + 1) - \log(p - 3) = 3$

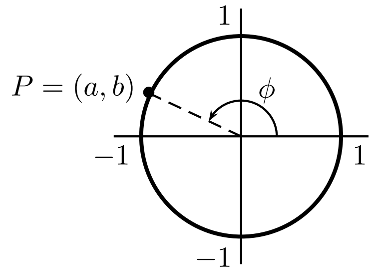
Solution: We apply a basic property of logarithms and then use the definition of the logarithm (or exponentiate).

$$\begin{aligned} \log(2p+1) - \log(p-3) &= 3 \\ \log\left(\frac{2p+1}{p-3}\right) &= 3 \\ \frac{2p+1}{p-3} &= 10^3 = 1000 \\ 2p + 1 &= 1000(p-3) = 1000p - 3000 \\ 3001 &= 998p \\ \frac{3001}{998} &= p \end{aligned}$$

Answer: $p = \frac{3001}{998}$

8. [10 points]

The point P (with coordinates (a, b)) is on the unit circle at angle ϕ, as shown in the diagram to the right. Use this information to **find the values below in terms of a and/or b.**
NOTE: Your answers should NOT include function names like "sin", "cos", or "tan".
You do not need to show your work for this problem.



a. [2 points] Find $\sin(\phi)$.

Solution: $\sin(\phi)$ is the y–coordinate of the point P, so $\sin(\phi) = b$.

Answer: $\sin(\phi) =$ b

b. [2 points] Find $\tan(-\phi)$.

Solution: The tangent function is an odd function, so

$$\tan(-\phi) = -\tan(\phi) = -\frac{\sin(\phi)}{\cos(\phi)} = -\frac{b}{a}.$$

Answer: $\tan(-\phi) =$ $-\frac{b}{a}$

c. [2 points] Find $\cos(\phi + \pi)$.

Solution: $\cos(\phi + \pi)$ is the x–coordinate of the point halfway around the circle from P, so $\cos(\phi + \pi) = - - a$. (Alternatively, note that the graph of $\cos(\phi + \pi)$ is the graph of $\cos(\phi + \pi)$ shifted left π units. This is the same as the graph of $-\cos(\phi)$, so $\cos(\phi + \pi) = -\cos(\phi) = -a$.)

Answer: $\cos(\phi + \pi) =$ $-a$

d. [2 points] Find $\sin(\phi - \frac{\pi}{2})$.

Solution: The graph of $\sin(\phi - \frac{\pi}{2})$ results from shifting the graph of $\sin(\phi)$ to the right $\pi/2$ units. This is the same as the graph of $-\cos(\phi)$, so $\sin(\phi - \frac{\pi}{2}) = -\cos(\phi) = -a$. (Alternatively, note that the point at angle $\phi - \pi/2$ has y–coordinate equal to the opposite of the x–coordinate of the point P.)

Answer: $\sin(\phi - \frac{\pi}{2}) =$ $-a$

e. [2 points] Find the coordinates of the point at angle ϕ on the circle of radius 7 centered at the point $(-3, 2)$.

Solution: The point at angle ϕ on the circle of radius 7 centered at the origin is $(7a, 7b)$, so the point at angle ϕ on the circle of radius 7 centered at the point $(-3, 2)$ is $(7a - 3, 7b + 2)$.

Answer: $(7a - 3, 7b + 2)$

9. [11 points] *For this problem, show your work step-by-step and give all answers in* **exact form** *or accurate to at least three decimal places. Include units.*

The concentration (in milligrams per milliliter) of a certain experimental medication ("Medication E") in a patient's bloodstream t hours after injection is $C(t) = De^{-1.5t}$, where D is the concentration immediately after the injection.

a. [2 points] By what percent does the concentration of Medication E in the bloodstream decrease each hour after injection?

Solution: The hourly decay factor is $e^{-1.5} \approx 0.22313$, so the concentration of Medication E decreases by a factor of $1 - e^{-1.5} \approx 0.77687$ or about 77.687% per hour.

Answer: **About** 77.687%

b. [3 points] What is the half-life of Medication E in the bloodstream?

Solution: Suppose h is the half-life of Medication E in the bloodstream. Then $C(h) = \frac{1}{2}D$ so $De^{-1.5h} = \frac{1}{2}D$ and $e^{-1.5h} = \frac{1}{2}$. Hence $-1.5h = \ln(1/2)$ so $h = \frac{\ln(1/2)}{-1.5} \approx 0.462$. Therefore, the half-life is approximately 0.462 hours.

Answer: $\dfrac{\ln(1/2)}{-1.5} \approx 0.462$ **hours**

Suppose that a patient is given two injections (Medications A and B) at the same time.

- Medication A has an initial blood concentration of 3 mg/ml, and its concentration decreases at a *continuous* hourly rate of 25%.
- Medication B has an initial blood concentration of 4.5 mg/ml, and its concentration decreases at a *continuous* hourly rate of 30%.

Let $A(t)$ and $B(t)$ be the blood concentration (in mg/ml) of Medication A and of Medication B, respectively, t hours after the patient receives these injections.

c. [2 points] Find a formula for $A(t)$ and a formula for $B(t)$.

$A(t) =$ $3e^{-0.25t}$ $B(t) =$ $4.5e^{-0.3t}$

d. [4 points] How long after the injections will the concentration of Medication B be only 2% more than the concentration of Medication A in the bloodstream?

Solution: The concentration of Medication B is 2% more than the concentration of Medication A when $B(t) = 1.02A(t)$, so we solve for t in this equation.

$$B(t) = 1.02A(t)$$
$$4.5e^{-0.3t} = 1.02(3e^{-0.25t}) = 3.06e^{-0.25t}$$
$$\ln(4.5e^{-0.3t}) = \ln(3.06e^{-0.25t})$$
$$\ln(4.5) + \ln(e^{-0.3t}) = \ln(3.06) + \ln(e^{-0.25t})$$
$$\ln(4.5) - 0.3t = \ln(3.06) - 0.25t$$
$$\ln(4.5) - \ln(3.06) = 0.05t \quad \text{so} \quad t = \frac{\ln(4.5) - \ln(3.6)}{0.05} = \frac{\ln(4.5/3.06)}{0.05} \approx 7.713.$$

Hence the concentration of Medication B is 2% more than the concentration of Medication A $\dfrac{\ln(4.5/3.06)}{0.05}$ or about 7.713 hours after the injections.

Answer: $\dfrac{\ln(4.5/3.06)}{0.05} \approx 7.713$ **hours**

10. [8 points] The management of a new pizza restaurant (called "New Pizza Restaurant" or "NPR") believes that the number of pizzas the restaurant will sell each month is a function of the amount of money it spends on advertising that month. Let $P(A)$ be the average number of pizzas the restaurant expects to sell in a month when it spends A dollars on advertising. Market research suggests that

$$P(A) = 600 + 50\ln(A+1).$$

Throughout this problem, show your work step-by-step and give all answers in **exact form** *or rounded accurately to at least two decimal places. Include units.*

a. [1 point] How many pizzas does NPR expect to sell in a month if they spend no money on advertising?

Solution: If no money is spent on advertising, then NPR expects to sell $P(0) = 600 + 50\ln(0+1) = 600 + 50\ln(1) = 600$ pizzas in a month.

Answer: 600 **pizzas**.

b. [3 points] How much money does NPR plan to spend on advertising in a month if they want to sell 1200 pizzas that month?

Solution: We solve for A in the equation $P(A) = 1200$.

$$\begin{aligned} P(A) &= 1200 \\ 600 + 50\ln(A+1) &= 1200 \\ 50\ln(A+1) &= 1200 - 600 = 600 \\ \ln(A+1) &= 600/50 = 12 \\ A+1 &= e^{12} \\ A &= e^{12} - 1 \approx 162,753.79 \end{aligned}$$

Hence NPR plans to spend about $\$162,754$ on advertising.

Answer: **about** $\$162,754$.

c. [4 points] A competitor, "OPR", expects to sell an average of $300 + 45\ln((A+1)^2)$ pizzas in a month when it spends A dollars on advertising. Suppose that in December, OPR and NPR will spend the same amount on advertising and will both expect to sell the same number of pizzas. How much will each restaurant spend on advertising in December?

Solution: We solve for A in the equation $300 + 45\ln((A+1)^2) = P(A)$.

$$\begin{aligned} 300 + 45\ln((A+1)^2) &= 600 + 50\ln(A+1) \\ 300 + 90\ln(A+1) &= 600 + 50\ln(A+1) \\ 40\ln(A+1) &= 300 \\ \ln(A+1) &= 7.5 \\ A+1 &= e^{7.5} \\ A &= e^{7.5} - 1 \approx 1807.04 \end{aligned}$$

Hence each restaurant will spend about $\$1807$ on advertising in December.

Answer: **about** $\$1807$.

11. [10 points] There is a third pizza restaurant in town known as "TPR". Let $Z(A)$ be the expected number of pizzas TPR will sell in a month when TPR spends A dollars on advertising. Let k be the average amount (in dollars) that TPR spends on advertising each month.

a. [2 points] Write an equation expressing the following statement:
"When TPR spends \$100 more than average on advertising, it expects to sell 25 more pizzas than average."

Answer: $Z(k + 100) = Z(k) + 25$

For each of the quantities in parts (b)–(e) below, pick the ONE expression from the list of "Answer Choices" that best represents the described quantity. Clearly write the CAPITAL LETTER of your choice on the answer blank provided.

Answer Choices for (b)-(e)

A. $Z(k)$	F. $Z(500k)$	K. $Z(k) - 0.5$	P. $Z^{-1}(k + 0.5)$
B. $Z^{-1}(k)$	G. $1.5Z(k)$	L. $Z(k) + 0.5$	Q. $Z^{-1}(1.5k)$
C. $Z(500)$	H. $0.5Z(k)$	M. $Z(k - 0.5)$	R. $Z^{-1}(1.5Z(k))$
D. $Z^{-1}(500)$	I. $Z(1.5k)$	N. $0.5Z^{-1}(k)$	S. $Z^{-1}(0.5k)$
E. $Z(k + 500)$	J. $Z(0.5k)$	O. $1.5Z^{-1}(k)$	T. $Z^{-1}(0.5Z(k))$

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b. [2 points] Which expression represents TPR's expected monthly pizza sales when \$500 is spent on advertising that month?

Answer: $Z(500)$ **(C)**

c. [2 points] Which expression represents the amount TPR spends on advertising in a month when it expects to sell 500 pizzas?

Answer: $Z^{-1}(500)$ **(D)**

d. [2 points] Which expression represents TPR's expected monthly pizza sales if it spends 50% less on advertising than average?

Answer: $Z(0.5k)$ **(J)**

e. [2 points] Which expression represents the amount TPR spends on advertising if it expects to sell 50% more pizzas than in a month with average spending on advertising?

Answer: $Z^{-1}(1.5Z(k))$ **(R)**