Math 105 — First Midterm

October 7, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	13	
3	10	
4	11	
5	13	
6	10	
7	8	
8	13	
9	10	
Total	100	

1. [12 points] Jerry Giraffe was a giraffe. He was six feet tall when he was born, and from that moment, he grew at a constant rate of three inches per month until he was twenty feet tall, at which point he stopped growing. He remained twenty feet tall for the rest of his life.

Recall that there are 12 inches in a foot and 12 months in a year.

a. [3 points] Let m be Jerry's age, in months, and let h be Jerry's height, in feet. Find a formula for h in terms of m that is valid during the time he was growing, that is, from the time Jerry was born until the time he reached his full-grown height of 20 feet.

Solution: Since h has a constant average rate of change, it is a linear function of m. The constant average rate of change is 0.25 feet per month and the initial value is 6 feet. This means a formula for h is given by $h = 6 + 0.25m$.

Answer: During the time that he was growing, $h =$ $6 + 0.25m$ **or** $6 + \dfrac{1}{4}m$

b. [2 points] How old was Jerry when he stopped growing, i.e. when he reached his full-grown height? *Include units.*

Solution: Recall that $h = 6 + 0.25m$. Setting this to his full-grown height, 20 feet, we find

$$\begin{aligned} 20 &= 6 + 0.25m \\ 14 &= 0.25m \\ 56 &= m \end{aligned}$$

So Jerry stopped growing when he was 56 months old.

Answer: **56 months old**

Let $j(m)$ be Jerry's height in feet when he was m months old. So $h = j(m)$.
Note that $j(m)$ is defined only while Jerry is alive.

c. [4 points] Jerry Giraffe died at the age of 400 months.
What are the domain and range of $j(m)$ in the context of this problem?
Use either interval notation or inequalities to give your answers.

Solution: Jerry was alive from the age of 0 to the age of 400 months, so the domain is the interval $[0, 400]$.
The range consists of all the values of Jerry's height during his life. He was born 6 feet tall and grew to a height of 20 feet. The range is thus the interval $[6, 20]$.

Answers: Domain: $[0, 400]$ Range: $[6, 20]$

d. [3 points] Give a formula for $j(m)$ in terms of m that is valid on its entire domain.
Hint: Use a piecewise-defined function.

Solution: Until Jerry reaches his maximum height at 56 months, the formula for his height is $6 + \frac{1}{4}m$. After he reaches his maximum height at 56 months, he is 20 feet tall until he dies at the age of 400 months.

Therefore, $j(m)$ has the formula $j(m) = \begin{cases} 6 + \frac{1}{4}m & \text{if } 0 \le m \le 56 \\ 20 & \text{if } 56 < m \le 400. \end{cases}$

Answer: $j(m) = \begin{cases} 6 + \frac{1}{4}m & \textbf{if } 0 \le m \le 56 \\ 20 & \textbf{if } 56 < m \le 400 \end{cases}$

2. [13 points] *Throughout this problem, remember to show your work carefully.*

a. [4 points] Find a formula for the quadratic function $g(x)$ described by the table below.

x	-4	1	2	7
$g(x)$	0	-5	-5	0

Solution: We see from the table that the zeros of g are $x = -4$ and $x = 7$. Therefore, a formula for $g((x)$ is given in factored form by $g(x) = a(x+4)(x-7)$ for some constant a. To find a, we use the fact that $g(1) = -5$, so $a(1+4)(1-7) = -5$. Then $-30a = -5$ so $a = \frac{-5}{-30} = \frac{1}{6}$. Thus $g(x) = \frac{1}{6}(x+4)(x-7)$ or, expanding to rewrite this in standard form, we have $g(x) = \frac{1}{6}x^2 - \frac{1}{2}x - \frac{14}{3}$.

Answer: $g(x) =$ $\frac{1}{6}(x+4)(x-7)$ **or** $\frac{1}{6}x^2 - \frac{1}{2}x - \frac{14}{3}$

b. [3 points] Given $f(x) = 13(x-8)^2 + w$, find the average rate of change of f from $x = 8$ to $x = 8 + h$. *Simplify your answer completely. Your answer may include h and/or w.*

Solution: The average rate of change of f from $x = 8$ to $x = 8 + h$ is given by

$$\frac{f(8+h) - f(8)}{(8+h) - 8} = \frac{\big(13((8+h)-8)^2 + w\big) - \big(13(8-8)^2 + w\big)}{h}$$
$$= \frac{(13h^2 + w) - (0 + w)}{h} = \frac{13h^2 + w - w}{h} = \frac{13h^2}{h} = 13h.$$

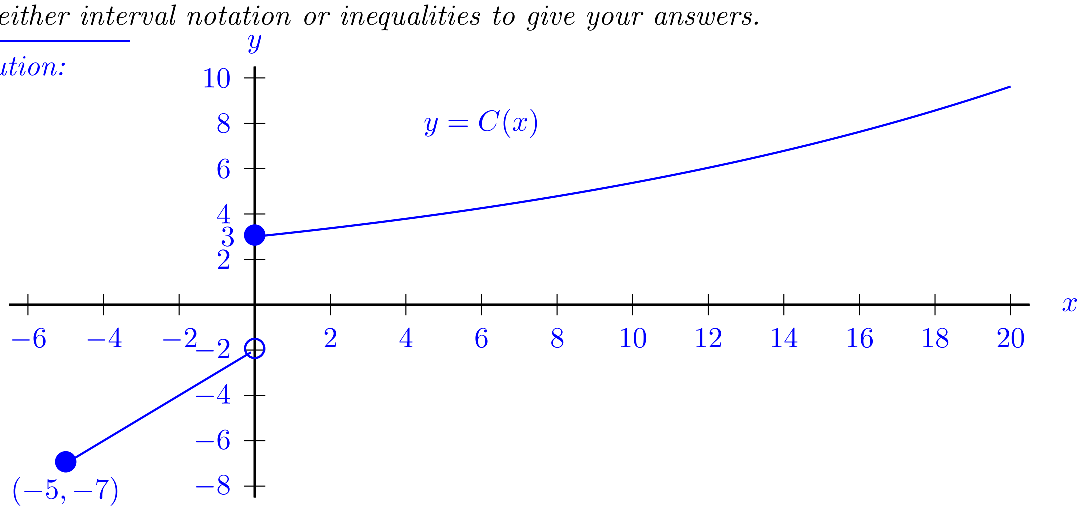
Answer: $13h$

c. [6 points] Consider the function C defined below.

$$C(x) = \begin{cases} -2 + x & \text{if } -5 \le x < 0 \\ 3(1.06)^x & \text{if } 0 \le x. \end{cases}$$

Sketch a graph of $y = C(x)$. Then find the domain and range of this function.
Remember to clearly label your axes.
Use either interval notation or inequalities to give your answers.

Solution:



Domain: $[-5, \infty)$ **Range:** $[-7, -2) \cup [3, \infty)$

3. [10 points] Annie Ant and Greta Grasshopper are having a debate about how to spend their time during October. Annie says that she will spend a total of 12 hours each day gathering food and building her anthill. Let B be the number of cm^3 of anthill that Annie builds in October, and let $D = g(B)$ be the number of grams of food that she gathers in October.

Annie knows that g is a linear function. She is also able to determine that if she builds 500 cm^3 of her anthill in October, then she will gather a total of 1500 grams of food but that if she builds only 150 cm^3 of her anthill, then she will gather a total of 2300 grams of food in October.

a. [4 points] Find a formula for $g(B)$.

Solution: Since g is linear, we first find its constant average rate of change. We have $g(500) = 1500$ and $g(150) = 2300$, so the constant average rate of change (slope) of g is

$$\frac{g(500) - g(150)}{500 - 150} = \frac{1500 - 2300}{350} = \frac{-800}{350} = -\frac{16}{7}.$$

Using point-slope form, we find $g(B) - 1500 = -\frac{16}{7}(B - 500)$ so

$$g(B) = 1500 - \frac{16}{7}(B - 500) = \frac{18500}{7} - \frac{16}{7}B.$$

Answer: $g(B) = 1500 - \frac{16}{7}(B - 500) = \frac{18500}{7} - \frac{16}{7}B$

b. [6 points] Find and interpret the slope and horizontal intercept of the graph of $D = g(B)$ in the context of this problem. *For each interpretation, remember to use a complete sentence and include units.*

Answers

Slope = $-\frac{16}{7}$ **grams of food per cm^3 of anthill**

Interpretation of slope: In October, for every 7 cm^3 of her anthill that Annie builds, she gathers 16 fewer grams of food.

Solution: The horizontal intercept occurs when $g(B) = 0$. Solving for B, we have $\frac{18500}{7} - \frac{16}{7}B = 0$ so $\frac{16}{7}B = \frac{18500}{7}$ and $B = \frac{18500}{16} = \frac{4625}{4} = 1156.25.$

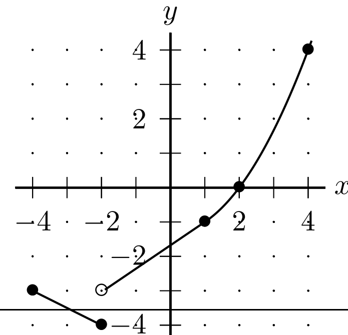
Horizontal intercept = $\frac{18500}{16} = \frac{4625}{4} = 1156.25$ **cm^3 of anthill**

Interpretation of horizontal intercept: In October, if Annie builds 1156.25 cm^3 of her anthill, she will gather no food.

4. [11 points] Invertible functions q, n, and h are described by the table, formula, and graph below. Use this information to answer the questions that follow.

x	-4	-1	0	1	4
$q(x)$	10	1	-1	-2	-4

$n(t) = 3 - 2t$



Graph of $y = h(x)$

a. [3 points] Based on the data in the table above, determine which of the following statements could be true about the function q on the entire interval from $x = -4$ to $x = 4$. (*Circle* ALL *such statements or circle* NONE OF THESE.)

q is an increasing function.

$\boxed{q \text{ is concave up.}}$

$\boxed{q \text{ is a decreasing function.}}$

q is concave down.

q is a linear function.

NONE OF THESE

b. [5 points] Evaluate each of the following.

(i) $h(-2) - 2q(-4)$

Solution: From the graph, we see $h(-2) = -4$. From the table, we see that $q(-4) = 10$. So

$h(-2) - 2q(-4) = -4 - 2(10) = -24.$

Answer: -24

(ii) $5q^{-1}(1)$

Solution: From the table, we see that $q^{-1}(1) = -1$. So

$$5q^{-1}(1) = 5(-1) = -5.$$

Answer: -5

(iii) $q(q(q(0)))$

Solution: From the table, we see that $q(0) = -1$, $q(-1) = 1$, and $q(1) = -2$. Thus

$q(q(q(0))) = q(q(-1)) = q(1) = -2.$

Answer: -2

(iv) $n(h^{-1}(-3))$

Solution: From the graph, we see that $h^{-1}(-3) = -4$. Hence

$n(h^{-1}(-3)) = n(-4) = 3 - 2(-4) = 11$

Answer: 11

c. [3 points] Find a formula for $4n(n(t))$. *Simplify your answer completely.*

Solution:

$$4n(n(t)) = 4n(3-2t) = 4(3-2(3-2t)) = 4(3-6+4t) = 4(-3+4t) = -12+16t$$

Answer: $4n(n(t)) =$ $16t - 12$

5. [13 points] Roo is a boxing kangaroo in Australia. Every Sunday, Roo has a boxing match against a professional boxer at the Sydney Opera House.

Let $r(t)$ be the revenue, in dollars, that the opera house makes from ticket sales when it sells t tickets to one of Roo's matches. Then

$$r(t) = t\left(230 - \frac{1}{30}t\right).$$

Note: The capacity of the Sydney Opera House is 5738, so there are never more than 5738 tickets sold to a match.

a. [5 points] If the opera house had a revenue of \$159,120 from ticket sales to last week's match, how many tickets did they sell? *Remember to show your work carefully.*

Solution: If the opera house has a revenue of \$159,120, then $r(t) = 159120$. We use the quadratic formula to solve for t in this equation.

$$\begin{aligned}159120 &= r(t) \\ 159120 &= t\left(230 - \frac{1}{30}t\right) \\ 159120 &= -\frac{1}{30}t^2 + 230t \\ 0 &= -\frac{1}{30}t^2 + 230t - 159120\end{aligned}$$

$$\begin{aligned}\text{So } t &= \frac{-230 \pm \sqrt{230^2 - 4(-\frac{1}{30})(-159120)}}{2(-\frac{1}{30})} \\ &= \frac{-230 \pm \sqrt{52900 - 21216}}{-\frac{1}{15}} \\ &= -15(-230 \pm \sqrt{31684}) \\ &= -15(-230 \pm 178) = 780 \text{ or } 6120\end{aligned}$$

Because the capacity of the opera house is 5738, the only valid solution is 780.

Answer: 780 **tickets**

b. [6 points] Use the method of completing the square to put the formula for $r(t)$ in vertex form. *Carefully show your algebraic work step-by-step.*

Solution:

$$\begin{aligned}r(t) &= t\left(230 - \frac{1}{30}t\right) = -\frac{1}{30}t^2 + 230t = -\frac{1}{30}(t^2 - 6900t) \\ &= -\frac{1}{30}\left[t^2 - 6900t + \left(\frac{-6900}{2}\right)^2 - \left(\frac{-6900}{2}\right)^2\right] \\ &= -\frac{1}{30}\left[(t-3450)^2 - (-3450)^2\right] \\ &= -\frac{1}{30}(t-3450)^2 + 396750\end{aligned}$$

Answer: $r(t) = -\frac{1}{30}(t-3450)^2 + 396750$

c. [2 points]

Solution: Using the vertex form we found above, the vertex is $(3450, 396750)$. Because the leading coefficient $(-\frac{1}{30})$ is negative, this gives the maximum value of the function. Thus the maximum revenue is \$396,750, and this occurs when 3450 tickets are sold.

What is the maximum possible revenue? $\$396,750$

How many tickets are sold to make the maximum possible revenue? 3450 **tickets**

6. [10 points] A local organic farm sells chicken eggs. Consider the following functions.

- $G(k)$ is the number of eggs produced in a day when the farm has k healthy chickens.
- $R(z)$ is the daily egg revenue (in dollars) the farm receives when it produces z eggs that day.

Throughout this problem, assume that the functions G and R are invertible.
For each of the sentences (a)-(e) below, fill in the blank with the one expression from the list of "possible answers" given below that makes the statement true.

No work or explanation is necessary for this problem.

Possible Answers:

10	$R^{-1}(10)$	$G(G(10))$	$G(R^{-1}(10))$
$G(10)$	$R(G(10))$	$R^{-1}(G(10))$	$R^{-1}(G^{-1}(10))$
$R(10)$	$G(R(10))$	$G^{-1}(R(10))$	
$G^{-1}(10)$	$R(R(10))$	$R(G^{-1}(10))$	$G^{-1}(R^{-1}(10))$

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a. [2 points]
If the farm produced 10 eggs today, then its daily egg revenue today was $\underline{\quad R(10) \quad}$ dollars.

b. [2 points]
If the farm produced 10 eggs today, then there were $\underline{\quad G^{-1}(10) \quad}$ healthy chickens.

c. [2 points]
Today the farm had 10 healthy chickens, so its daily egg revenue was $\underline{\quad R(G(10)) \quad}$ dollars.

d. [2 points]
If the farm produced $R^{-1}(10)$ eggs today, then its daily egg revenue was $\underline{\quad 10 \quad}$ dollars.

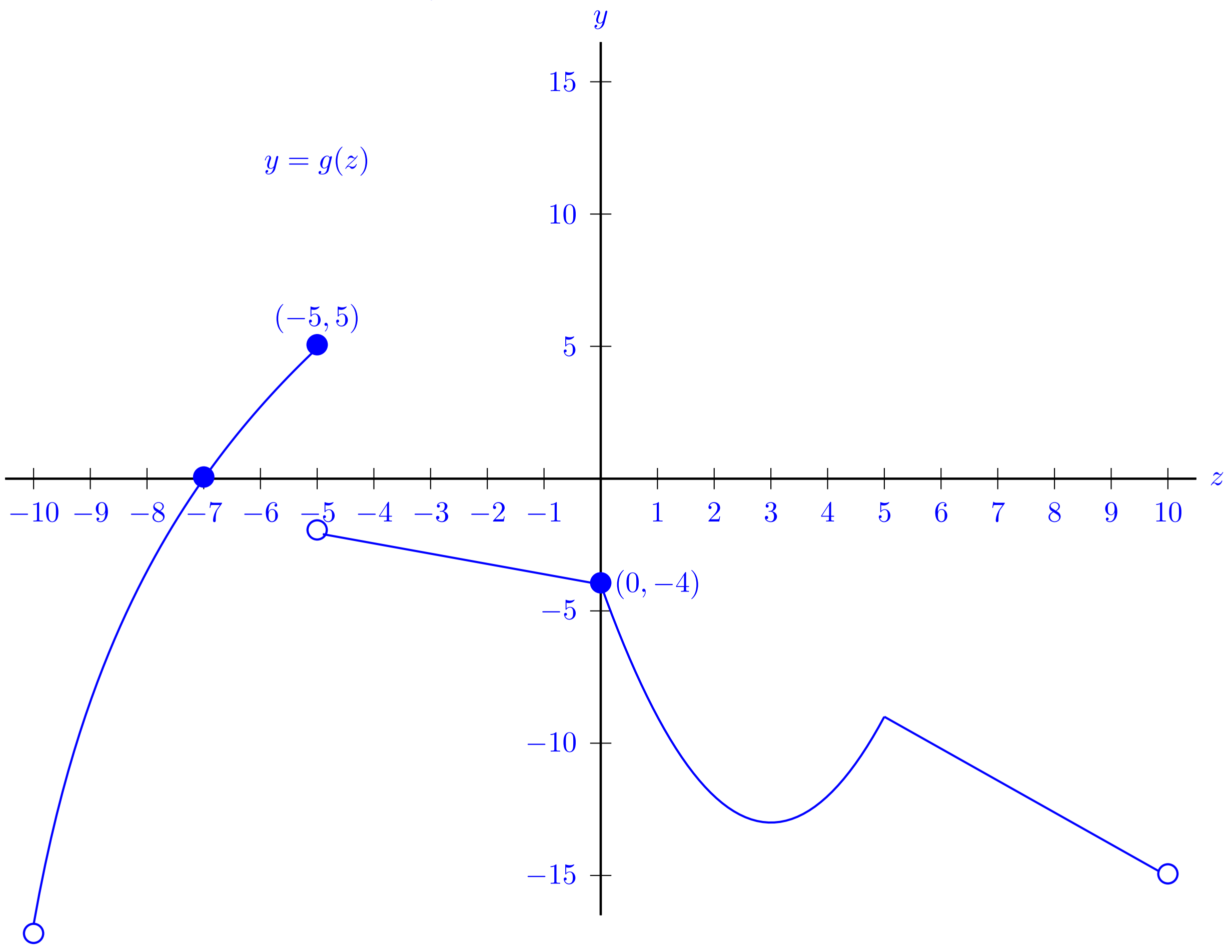
e. [2 points]
If the farm's daily egg revenue today was \$10, then there were $\underline{\quad G^{-1}(R^{-1}(10)) \quad}$ healthy chickens.

7. [8 points] On the axes provided below, sketch the graph of **one function** $y = g(z)$ satisfying all of the following:

- The domain of $g(z)$ is the interval $(-10, 10)$.
- The range of $g(z)$ includes the number 5.
- $g(0) = -4$ and $g(-7) = 0$
- $g(z)$ has exactly one zero.
- $g(z)$ is increasing and concave down for $-10 < z < -5$.
- $g(z)$ is decreasing for $-5 < z < 0$.
- $g(z)$ has a constant average rate of change for $5 < z < 10$.

Please make sure that your sketch is large, well-labeled, and unambiguous.

Solution: Note that there are many possible solutions.



8. [13 points] Roger the rabbit is a large rabbit that likes to eat! On a normal day, Roger has a daily meal of 12 ounces of carrots and 7 ounces of lettuce mixed together. However, sometimes Roger will want to eat a different mix for his daily meal. Let $R(z)$ be the ratio of the amount of lettuce in his food mix to the total amount of food if $|z|$ ounces of lettuce have been added ($z > 0$) or removed ($z < 0$). Note that Roger starts with 12 ounces of carrots and 7 ounces of lettuce and that the amount of carrots does NOT change.

a. [3 points] Evaluate $R(0)$, $R(4)$ and $R(-0.5)$.

Solution: $R(0)$ is the initial ratio of lettuce to food mix. So $R(0) = \frac{7}{7+12} = \frac{7}{19} \approx 0.3684$. $R(4)$ is the ratio of lettuce to food mix when he adds 4 ounces of lettuce to his food mix. Thus $R(4) = \frac{7+4}{7+12+4} = \frac{11}{23} \approx 0.47826$. Finally $R(-0.5)$ is the ratio of lettuce to food mix when he takes 0.5 ounces of lettuce out of his food mix. Hence $R(-0.5) = \frac{7-0.5}{7+12-0.5} = \frac{6.5}{18.5} = \frac{13}{37} \approx 0.35135$.

$R(0) =$ $\frac{7}{19}$ $\qquad R(4) =$ $\frac{11}{23}$ $\qquad R(-0.5) =$ $\frac{13}{37}$

b. [4 points] Find the domain and range of $R(z)$ in the context of this problem.
Use either inequalities or interval notation to express your answers.

Solution: Roger cannot take more than 7 ounces of lettuce out of his mixture, but he can add as much lettuce as he wants to.[1] This means that the domain is $[-7, \infty)$. If he removes all 7 ounces of lettuce from his food mix, the ratio is 0. If he adds as much lettuce as possible, the ratio will be close to 1, but never attain this value. This means that the range is $[0, 1)$.[1]

Domain: $[-7, \infty)$ **Range:** $[0, 1)$

c. [2 points] Find a formula for $R(z)$ in terms of z.

Solution: When Roger adds z ounces of lettuce, there are $7 + z$ ounces of lettuce in the mixture and there are $12 + 7 + z$ total ounces of food mixture. Thus, the resulting ratio of the amount of lettuce in his food mix to the total amount of food mix is $R(z) = \dfrac{7+z}{12+7+z} = \dfrac{7+z}{19+z}$.

Answer: $R(z) = \dfrac{7+z}{19+z}$

d. [4 points] If Roger wants a food mixture with 65% lettuce, how much lettuce must he add or remove to create this mixture? *Show your work carefully, round to the nearest* 0.1 *ounce, include units, and clearly indicate whether lettuce should be added or removed.*

Solution: We want to find z so that $R(z) = 0.65$, so we solve for z in the equation $\dfrac{7+z}{19+z} = 0.65$.

$$\frac{7+z}{19+z} = 0.65 \qquad z - 0.65z = 12.35 - 7$$

$$7 + z = 0.65(19+z) \qquad 0.35z = 5.35$$

$$7 + z = 12.35 + 0.65z \qquad z = \frac{5.35}{0.35} = \frac{107}{7} \approx 15.2857$$

Answer: $\dfrac{107}{7}$ **or about** 15.3 **ounces of lettuce must be added**

[1]Note that there technically is a maximum amount of lettuce Roger could add due to the available supply of lettuce, so the domain is $[-7, L_{max}]$, where L_{max} is the maximum number of ounces of lettuce actually available to be added. If we consider this restriction on the lettuce supply, then the range is instead $\left[0, \frac{7+L_{max}}{19+L_{max}}\right]$.

9. [10 points] Annie Ant finished building her anthill, and it immediately started eroding because of the weather. Every day, her anthill loses 1.5% of its volume. Let $v(d)$ be the volume, in cm^3, of Annie's anthill d days after she finished building it. Assume that her anthill was 1200 cm^3 when she finished building it.

a. [2 points] Based on the description above, answer each of the following questions. In each case, *circle the one best answer.* Note: You do NOT need to explain your reasoning.

Solution: According to the description, the volume of the anthill changes at a constant percent rate per unit time, so $v(d)$ is an exponential function. Since the anthill is losing volume, $v(d)$ is a decreasing function. (The volume of the anthill is decaying exponentially.)

(i) What kind of function is $v(d)$?

○ linear ○ quadratic ○ $\boxed{\text{exponential}}$ ○ NONE OF THESE

(ii) Which of the following accurately describes $v(d)$?

○ $v(d)$ is an increasing function. ○ $\boxed{v(d) \text{ is a decreasing function.}}$

○ NEITHER OF THESE

b. [3 points] Find a formula for $v(d)$ in terms of d.

Solution: The initial value is 1200. Because the anthill loses 1.5% of its volume every day, the decay factor of this exponential function is $1 - 0.015 = 0.985$. So a formula for the exponential function $v(d)$ is

$$v(d) = 1200(0.985)^d$$

Answer: $v(d) =$ $1200(0.985)^d$

c. [3 points] Give a practical interpretation of the expression $v^{-1}(50)$ in the context of this problem. *Use a complete sentence and include units. Note that you do not need to evaluate* $v^{-1}(50)$.

Solution: After the anthill is finished being built, it takes $v^{-1}(50)$ days for the volume of the anthill to be 50 cm^3.

d. [2 points] Solve for a in the equation $v^{-1}(a) = 10$. *Either give your answer in exact form or rounded to the nearest* 0.01.

Solution: If $v^{-1}(a) = 10$, then $v(10) = a$. Therefore

$$a = v(10) = 1200(0.985)^{10} \approx 1031.67653071.$$

Answer: $a = 1200(0.985)^{10}$ **or** $a \approx 1031.68$