Math 105 — Final Exam

December 17, 2013

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
LA Post-Test	5	
1	10	
2	7	
3	10	
4	13	
5	5	
6	5	
7	8	
8	11	
9	5	
10	4	
11	7	
12	10	
Total	100	

1. [10 points] Foghorn is a chicken that is learning how to fly. In fact, he trains every day by jumping off the top of his coop and flapping his wings. Today, his height above the ground, in feet, t seconds after jumping is given by the function $h(t) = -16t^2 + 20t + 6$.
Note that once he lands on the ground, he stays on the ground.

a. [2 points] How long after Foghorn jumps off his coop does he hit the ground?
Be sure to show your work and give your final answer in exact form.

Solution: To find when Foghorn hits the ground, we set $h(t) = 0$ and solve for t. Using the quadratic formula, we find

$$t = \frac{-20 \pm \sqrt{20^2 - 4(-16)(6)}}{2(-16)} = \frac{-20 \pm \sqrt{784}}{-32} = \frac{-20 \pm 28}{-32}$$

So, $t = -0.25$ and $t = 1.5$. Because he starts on top of the coop, $t = -0.25$ does not make any sense as an answer. So, Foghorn hits the ground 1.5 seconds after jumping off the coop.

Answer: 1.5 **seconds**

b. [4 points] Use the method of completing the square to put the formula for $h(t)$ into vertex form. *Carefully show your algebraic work step-by-step.*

Solution:

$$h(t) = -16t^2 + 20t + 6 = -16\left(t^2 - \frac{5}{4}t\right) + 6 = -16\left(t^2 - \frac{5}{4}t + \left(\frac{5}{8}\right)^2 - \left(\frac{5}{8}\right)^2\right) + 6$$

$$= -16\left(\left(t - \frac{5}{8}\right)^2 - \left(\frac{5}{8}\right)^2\right) + 6 = -16\left(t - \frac{5}{8}\right)^2 + 16\left(\frac{5}{8}\right)^2 + 6 = -16\left(t - \frac{5}{8}\right)^2 + 12.25$$

Answer: $h(t) =$ $-16\left(t - \frac{5}{8}\right)^2 + 12.25$

c. [2 points] What is the maximum height Foghorn reaches? **Answer:** 12.25 **feet**

When does he reach his maximum height? **Answer:** 5/8 **seconds after he jumps**

Solution: The vertex form gives a maximum height of 12.25 feet 5/8 seconds after he jumps.

d. [2 points] What are the domain and range of $h(t)$ in the context of this problem?
Use either interval notation or inequalities to give your answers.

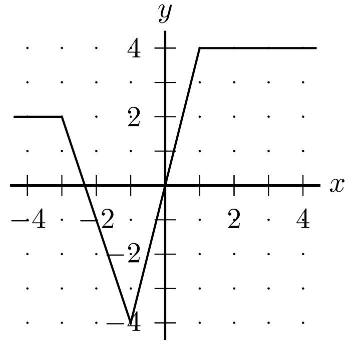
Solution: In the context of this problem, the domain is from when he jumps to when he lands. The domain is then $[0, 1.5]$. The range is from the ground, 0 feet, to the maximum height he reaches, which is 12.25 feet. The range is then $[0, 12.25]$.

Answers: Domain: $[0, 1.5]$ Range: $[0, 12.25]$

2. [7 points] Invertible functions f and g and a function h are described by the table, formula, and graph below. Use this information to answer the questions that follow.

x	-4	-3	-2	-1	0	1	2	3	4
$f(x)$	3	-2	1	4	-3	0	-4	2	-1

$$g(t) = \begin{cases} 4+x & \text{if } x < 3 \\ 2^x & \text{if } x \geq 3 \end{cases}$$



Graph of $y = h(x)$

Evaluate each of the following quantities, if possible.
If the specified quantity is undefined, write "UNDEFINED".
You do not have to show your work. However, any work you show may be worth partial credit.

a. [1 point] $f(0)h(-4)$

Solution: $f(0) = -3$ and $h(-4) = 2$. So, $f(0)h(-4) = (-3)(2) = -6$.

Answer: -6

b. [1 point] $3f(g(-2))$

Solution: $g(-2) = 4 + (-2) = 2$. So $3f(g(-2)) = 3f(2) = 3(-4) = -12$.

Answer: -12

c. [1 point] $f^{-1}(h(1) - 2)$

Solution:
$h(1) = 4$ so $h(1) - 2 = 4 - 2 = 2$.
Hence $f^{-1}(h(1) - 2) = f^{-1}(2) = 3$.

Answer: 3

d. [1 point] $g^{-1}(4)$

Solution: g is invertible and piecewise-defined, so we must figure out which part of the piecewise formula has an output of 4. 2^x is defined for $x \geq 3$, so this piece gives outputs in the interval $[8, \infty)$. So, we must solve

$$4 + x = 4.$$

The solution is $x = 0$. So $g^{-1}(4) = 0$.

Answer: 0

e. [1 point] $g(g(-1))$

Solution: $g(-1) = 4 + (-1) = 3$. So $g(g(-1)) = g(3) = 2^3 = 8$.

Answer: 8

f. [1 point] $k(-1)$ if $k(x) = \frac{1}{3}h(3x)$

Solution:

$$k(-1) = \frac{1}{3}h(3(-1)) = \frac{1}{3}h(-3) = \frac{1}{3}(2) = \frac{2}{3}$$

Answer: $2/3$

g. [1 point] Find the average rate of change of $h(x)$ between $x = -1$ and $x = 4$.

Solution: This average rate of change is given by

$$\frac{h(4) - h(-1)}{4 - (-1)} = \frac{4 - (-4)}{5} = \frac{8}{5}.$$

Answer: $8/5$

3. [10 points] Let $G(v)$ be the number of minutes it takes Goober the gorilla to eat a meal consisting of v pounds of vegetation.

a. [2 points] Suppose b and n are positive constants.
Give a practical interpretation of the equation $G^{-1}(b) = n$ in the context of this problem. *Use a complete sentence and include units.*

Solution: It takes Goober b minutes to eat a meal consisting of n pounds of vegetation.

b. [4 points] Suppose that there are positive constants c and d so that a formula for $G(v)$ is given by

$$G(v) = cv^d.$$

If $G(2) = 9$ and $G(3) = 18$, find the *exact* values of the constants c and d.

Solution: $G(2) = 9$ and $G(3) = 18$, so $c(2)^d = 9$ and $c(3)^d = 18$. Taking ratios, we find

$$\frac{c(3)^d}{c(2)^d} = \frac{18}{9}$$

$$\frac{3^d}{2^d} = 2$$

$$\left(\frac{3}{2}\right)^d = 2$$

$$1.5^d = 2$$

Using logarithms: $\ln(1.5^d) = \ln(2)$

$$d\ln(1.5) = \ln(2)$$

$$d = \frac{\ln(1.5)}{\ln(2)}$$

Substituting into the equation $c(2)^d = 9$, we find $c(2)^{\ln(1.5)/\ln(2)} = 9$ so $c = \dfrac{9}{2^{\ln(1.5)/\ln(2)}}$.

Answers: $c =$ $\dfrac{9}{2^{\ln(1.5)/\ln(2)}}$ and $d =$ $\dfrac{\ln(1.5)}{\ln(2)}$

c. [4 points] Suppose that the number of minutes it takes Goober's friend Toober to eat a meal consisting of v pounds of vegetation is $m = T(v)$, which is given by the formula

$$T(v) = q + \frac{\ln(v+2)}{\ln(5)}$$

for some constant q. Find a formula for $T^{-1}(m)$. *Show your work carefully.*
Note that your answer should be in exact form and be given in terms of m and q.

Solution: To find $T^{-1}(m)$, we solve for v in the equation $m = q + \dfrac{\ln(v+2)}{\ln(5)}$.

$$m = q + \frac{\ln(v+2)}{\ln(5)}$$

$$m - q = \frac{\ln(v+2)}{\ln(5)}$$

$$(\ln(5))(m-q) = \ln(v+2)$$

Exponentiating: $e^{\ln(5)(m-q)} = v + 2$

$$e^{\ln(5)(m-q)} - 2 = v$$

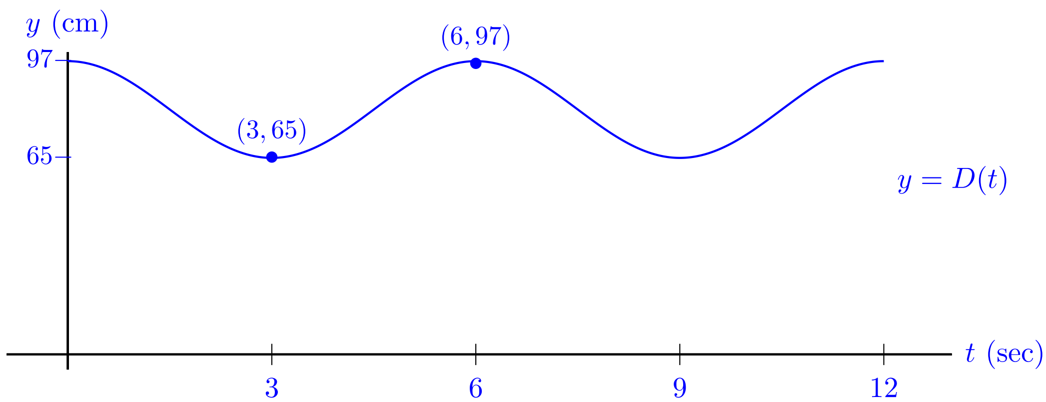
Thus $T^{-1}(m) = e^{\ln(5)(m-q)} - 2$.

Note that $e^{\ln(5)(m-q)} = \left(e^{\ln(5)}\right)^{m-q} = 5^{m-q}$ so we can simplify to $T^{-1}(m) = 5^{m-q} - 2$.

Answer: $T^{-1}(m) =$ $5^{m-q} - 2$ **(or** $e^{\ln(5)(m-q)} - 2$**)**

4. [13 points] Severus Snake is slithering along the banks of a river. At noon, a scientist starts to track Severus's distance away from the edge of the river. After a few minutes, the scientist realizes Severus's distance away from the edge of the river is a sinusoidal function. Let $D(t)$ be Severus's distance, in centimeters, away from the edge of the river t seconds after noon.

a. [5 points] At noon exactly, the scientist notes that Severus is 97 centimeters away from the edge of the river, which is the farthest away he ever gets. Three seconds after that, Severus's distance is 65 centimeters away from the river, the closest he gets.
Graph $y = D(t)$ for $0 \le t \le 12$. (*Clearly label the axes and important points on your graph. Be very careful with the* **shape and key features of your graph.**)



b. [6 points] Find the period, amplitude, equation of the midline, and a formula for the sinusoidal function $D(t)$. (*Include units for the period and amplitude.*)

Solution: A maximum and succeeding minimum of a sinusoidal function occur half a period apart. In this case, they occur 3 seconds apart, so the period is 6 seconds.
The midline is found by averaging the maximum and minimum values. This gives $y = \frac{97+65}{2} = 81$.
The amplitude is the distance between a maximum or minimum and the midline which is $|97 - 81| = 16$ centimeters.
Because the maximum occurs at noon ($t = 0$), the cosine function is a good candidate to use here. Let $D(t) = A\cos(B(t-h)) + k$. Then we have $A = 16$, $B = \frac{2\pi}{6} = \frac{\pi}{3}$, and $k = 81$. So, we have $D(t) = 16\cos\left(\frac{\pi}{3}(t-h)\right) + 81$. Because the maximum occurs at $t = 0$, we do not need a horizontal (or phase) shift. Thus, one possible formula for $D(t)$ is $D(t) = 16\cos\left(\frac{\pi t}{3}\right) + 81$.

Period: 6 **seconds**

Amplitude: 16 **centimeters**

Midline: $y = 81$

Formula: $D(t) = 16\cos(\frac{\pi}{3}t) + 81$

c. [2 points] How far away from the river is Severus 11 seconds after noon?
Give your answer accurate to at least two decimal places.

Solution: Severus is $D(11)$ centimeters away from the river 11 seconds after noon. Using the formula from above, $D(11) = 16\cos\left(\frac{11\pi}{3}\right) + 81 = 16\left(\frac{1}{2}\right) + 81 = 8 + 81 = 89$. So, Severus is 89 centimeters away from the river 11 seconds after noon.

Answer: 89 **centimeters**

5. [5 points] Find a formula for *one* polynomial $p(z)$ that satisfies *all* of the following conditions:

- $\lim\limits_{z\to\infty} p(z) = -\infty$ and $\lim\limits_{z\to-\infty} p(z) = -\infty$
- The only zeros of $p(z)$ are $z = -2$, $z = 1$, and $z = 3$.
- The point $(2, -12)$ is on the graph of $p(z)$.
- The degree of $p(z)$ is at most 5.

Show your work and reasoning carefully. You might find it helpful to first sketch a graph. There may be more than one possible answer, but you should give only one answer.

Solution: Because $\lim\limits_{z\to\infty} p(z) = -\infty$ and $\lim\limits_{z\to-\infty} p(z) = -\infty$, we know that the degree of $p(z)$ is even and the leading coefficient is negative.
Because $p(z)$ has zeros at $z = -2$, $z = 1$, and $z = 3$, the degree of the polynomial must be at least 3. Since it has to be of even degree less than 5, the degree must be 4. Thus there must be exactly one double zero.
Since the point $(2, -12)$ is on the graph, we can see by sketching the graph that the double zero must be at $z = 3$.
So, we have $p(z) = a(z+2)(z-1)(z-3)^2$ for some negative constant a.
Since $p(2) = -12$ we find $-12 = p(2) = a(2+2)(2-1)(2-3)^2 = a(4)(1)(-1)^2 = 4a$ so $a = -3$.
Thus, $p(z) = -3(z+2)(z-1)(z-3)^2$.

Answer: $p(z) =$ $-3(z+2)(z-1)(z-3)^2$

6. [5 points] Find all solutions to the equation

$$5\tan\left(2x + \frac{\pi}{2}\right) - 13 = 12$$

for x between 0 and 5. *Show your work carefully and give your answer(s) in exact form.*

Solution:

$$5\tan\left(2x + \frac{\pi}{2}\right) - 13 = 12 \quad \text{so} \quad 5\tan\left(2x + \frac{\pi}{2}\right) = 25 \quad \text{and} \quad \tan\left(2x + \frac{\pi}{2}\right) = 5.$$

Using the inverse tangent function, we find that one solution to the equation is given by

$$2x + \frac{\pi}{2} = \arctan(5) \quad \text{so} \quad 2x = \arctan(5) - \frac{\pi}{2} \quad \text{and} \quad x = \frac{\arctan(5) - \frac{\pi}{2}}{2}.$$

Note that the period of $5\tan\left(2x + \frac{\pi}{2}\right) - 13$ is $\pi/2$ and that the solution $\frac{\arctan(5) - \frac{\pi}{2}}{2}$ is *not* in the interval $[0, 5]$. Other solutions to the equation are obtained by adding integer multiples of the period $\pi/2$ to $\frac{\arctan(5) - \frac{\pi}{2}}{2}$. The resulting solutions in the interval $[0, 5]$ are

$$\frac{\arctan(5) - \frac{\pi}{2}}{2} + \frac{\pi}{2}, \quad \frac{\arctan(5) - \frac{\pi}{2}}{2} + \pi, \quad \text{and} \quad \frac{\arctan(5) - \frac{\pi}{2}}{2} + \frac{3\pi}{2}.$$

These can be simplified to $\frac{\arctan(5)}{2} + \frac{\pi}{4}$, $\frac{\arctan(5)}{2} + \frac{3\pi}{4}$, and $\frac{\arctan(5)}{2} + \frac{5\pi}{4}$.

Answer: $x =$ $\frac{\arctan(5)}{2} + \frac{\pi}{4}$, $\frac{\arctan(5)}{2} + \frac{3\pi}{4}$, **and** $\frac{\arctan(5)}{2} + \frac{5\pi}{4}$

7. [8 points] Freckles and Comet are cats in the same household. Consider the functions F, C, and D which are defined as follows:

- $F(m)$ is the number of ounces of food that Freckles eats in month m.
- $C(m)$ is the number of ounces of food that Comet eats in month m.
- $D(q)$ is the cost of buying q cans of cat food at a time when there are no sale prices.

Assume that D is invertible.
For each of questions below, circle the ONE best answer from among the options provided.
If none of the options are correct, circle NONE OF THESE.
Please note: To receive credit, you must clearly circle your choices. (Circle the **entire** *answer. If there is any ambiguity in your answer, you will not receive credit.)*

a. [1 point] What is the total number of ounces of food that Freckles and Comet eat in month m?

$D(m)+C(m)$ $\qquad \boxed{F(m)+C(m)}$ $\qquad F(C(m))$ $\qquad C(F(m))$ $\qquad$ NONE OF THESE

Solution: Freckles eats $F(m)$ ounces of food in month m. Comet eats $C(m)$ ounces of food in month m. Combined, they eat $F(m)+C(m)$ ounces of food in month m.

b. [2 points] Suppose that there are 3 ounces of food per can. What is the total cost of the food Freckles eats in month 4?

$\dfrac{D(4)}{3}$ $\qquad 3D(4)$ $\qquad D(3F(4))$ $\qquad \boxed{D\left(\dfrac{F(4)}{3}\right)}$ $\qquad$ NONE OF THESE

Solution: If Freckles eats $F(4)$ ounces of food in month 4 and there are 3 ounces of cat food per can, Freckles eats $\frac{F(4)}{3}$ cans of cat food in month 4. The cost of this is $D\left(\frac{F(4)}{3}\right)$.

c. [1 point] Let $A(q)$ be the average cost per can of buying q cans of cat food. Which of the following is a formula for $A(q)$?

$D^{-1}(q)$ $\qquad \dfrac{q}{D(q)}$ $\qquad \dfrac{F(q)+C(q)}{2}$ $\qquad \boxed{\dfrac{D(q)}{q}}$ $\qquad$ NONE OF THESE

Solution: The average cost per can of buying q cans of cat food is the cost of buying q cans of cat food, $D(q)$, divided by the number of cans of cat food purchased, q.

d. [1 point] When there are no sale prices, how many cans of cat food can be purchased at a time for \$20?

$D(20)$ $\qquad \boxed{D^{-1}(20)}$ $\qquad 20D^{-1}(q)$ $\qquad \dfrac{1}{D(20)}$ $\qquad$ NONE OF THESE

Solution: If we want to spend \$20 on cans of cat food, we want to solve for q in the equation $D(q)=20$. Thus, we can buy $D^{-1}(20)$ cans of cat food for \$20.

e. [2 points] Suppose that Comet eats at least twice as much food each month as Freckles eats. Which one of the following inequalities most accurately describes this relationship?

$C(m)\le 2F(m)$ $\qquad \boxed{C(m)\ge 2F(m)}$ $\qquad 2C(m)\le F(m)$ $\qquad 2C(m)\ge F(m)$

Solution: Twice as much food as Freckles eats in month m is $2F(m)$. Comet eats $C(m)$ ounces of food in month m, so the relationship is that $C(m)\ge 2F(m)$ for all m.

f. [1 point] If cat food goes on sale for 40% off its regular price, what is the cost of buying 20 cans of cat food at one time?

$\boxed{0.6D(20)}$ $\qquad 1.4D(20)$ $\qquad 0.4D(20)$ $\qquad D(8)$ $\qquad$ NONE OF THESE

Solution: If cat food is at its regular price, the cost of buying 20 cans of cat food is $D(20)$. If we take 40% off the regular price, we have $D(20)-0.4D(20)=0.6D(20)$.

8. [11 points] On the beaches of Mexico, there is a population of picky snails that wait for special shells to wash up onto the shore. These snails can only live in these particular shells, as the snails have become accustomed to the comfort in these shells.
Suppose the number of hundreds of special shells on the beaches of Mexico t years after the beginning of 2013 is

$$h(t) = (t^2 + 7)(4t - 7)^2$$

and the population, in hundreds, of picky snails t years after the beginning of 2013 is

$$p(t) = (t - 2)^2(8t^2 + 30).$$

Throughout this problem, remember to clearly show your work and reasoning.

a. [3 points] Find the leading term and any zeros of $h(t)$. If appropriate, write "NONE" in the answer blank provided.

Solution: To find the zeros of $h(t)$, we set $h(t) = 0$ and solve for t. Because $t^2 + 7$ is always positive, $h(t) = 0$ when $(4t - 7)^2 = 0$. So the zeros of $h(t)$ occur at $t = 7/4$.
To find the leading term, we take the product of the leading terms of its factors:

$$\text{leading term} = (t^2)(4t)(4t) = 16t^4$$

Answers: Leading Term: $16t^4$ Zero(s): $t = 7/4$

b. [3 points] The number of shells per snail is $Q(t) = \dfrac{h(t)}{p(t)}$.

Find the equations of all vertical asymptotes ("V.A.") and horizontal asymptotes ("H.A.") of the graph of $y = Q(t)$. If appropriate, write "NONE" in the answer blank provided.

Solution: To find vertical asymptotes, since $h(t)$ and $p(t)$ have no common factors, we set $p(t) = 0$ and solve for t. Because $8t^2 + 30$ is always positive, $p(t) = 0$ only when $(t - 2)^2 = 0$. So the only zero of $p(t)$ is $t = 2$. So, since $h(t) \neq 0$, the only vertical asymptote of $Q(t)$ is $t = 2$.
To find horizontal asymptotes, we look at the long-run behavior of $Q(t)$.

$$\lim_{t\to\infty} \frac{h(t)}{p(t)} = \lim_{t\to\infty} \frac{(t^2+7)(4t-7)^2}{(t-2)^2(8t^2+30)} = \lim_{t\to\infty} \frac{16t^4}{8t^4} = 2$$

$$\lim_{t\to-\infty} \frac{h(t)}{p(t)} = \lim_{t\to-\infty} \frac{(t^2+7)(4t-7)^2}{(t-2)^2(8t^2+30)} = \lim_{t\to-\infty} \frac{16t^4}{8t^4} = 2$$

Thus, the horizontal asymptote of $Q(t)$ is $y = 2$.

Answers: V.A.: $t = 2$ H.A.: $y = 2$

There is a competitive population of crabs that live on the same beaches. Suppose that there are 1200 of these crabs at the beginning of 2013, and that the population grows at a *continuous annual rate* of 35%. Let $c(t)$ be the population, in hundreds, of these crabs t years after the beginning of 2013.

c. [2 points] Find a formula for $c(t)$.

Solution: If $c(t) = Pe^{kt}$ is the population, in hundreds, of these crabs t years after the beginning of 2013, then $c(0) = 12$, so $P = 12$. With a continuous annual growth rate of 35%, we know that

$$c(t) = 12e^{.35t}.$$

Answer: $c(t) = 12e^{.35t}$

d. [3 points] The crabs like the same special shells as the snails do. Write a formula for the ratio of the number of shells to the number of crabs t years after the beginning of 2013.

Answer: $\dfrac{h(t)}{c(t)} = \dfrac{(t^2+7)(4t-7)^2}{12e^{.35t}}$

In the long run, what happens to the ratio of the number of shells to the number of crabs? In other words, assuming the functions described in this problem continue to be accurate models, what happens to this ratio after many, many years?
You must clearly indicate your reasoning in order to receive any credit for this problem.

Solution: The ratio of number of shells to the number of crabs t years after the beginning of 2013 is given by

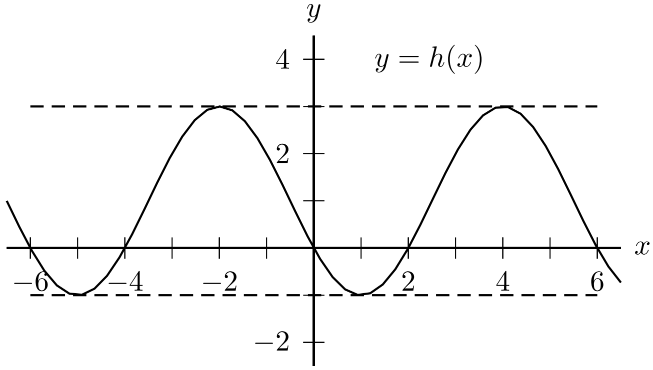
$$R(t) = \frac{h(t)}{c(t)}$$

To describe the ratio after many, many years, we look at the long-run behavior. Because we are looking at years after 2013, we only look at the limit of $R(t)$ as t grows without bound, i.e. as $t \to \infty$. Since exponential growth eventually dominates polynomial growth, the limit is zero. That is

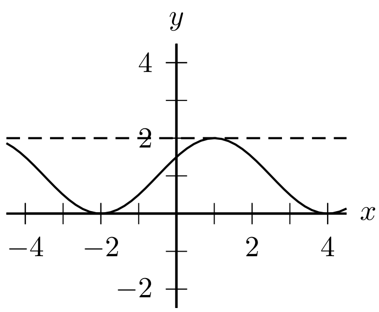
$$\lim_{t\to\infty} R(t) = \lim_{t\to\infty} \frac{h(t)}{c(t)} = \lim_{t\to\infty} \frac{(t^2+7)(4t-7)^2}{12e^{.35t}} = \lim_{t\to\infty} \frac{16t^4}{12e^{.35t}} = 0$$

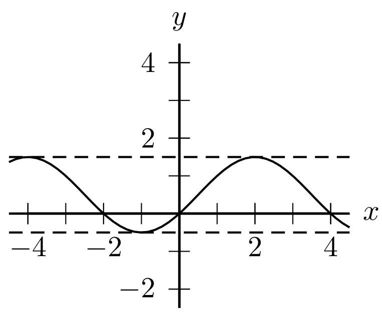
This means that in the long run, the ratio of the number of shells to the number of crabs approaches zero.

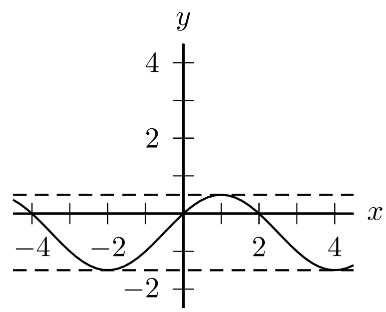
9. [5 points] Note that throughout this problem, you are not required to show your work. A portion of the graph of a sinusoidal function $h(x)$ is shown below.



a. [2 points] Which, if any, of the figures below shows part of the graph of $y = -\frac{1}{2}h(x)$? *Note that the scale is smaller than in the original graph above. Be sure to pay attention to the scale indicated on the axes.*


(Option A)


(Option B)

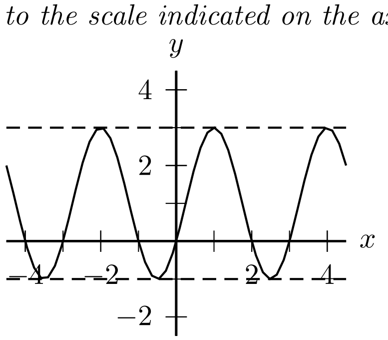

(Option C)

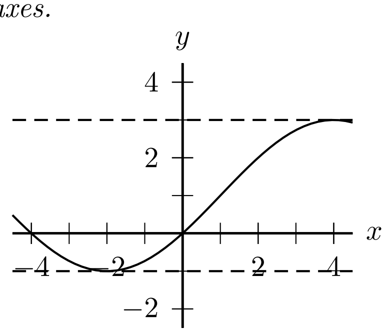
Circle your one final answer below. (Only the answer you circle below will be graded.)

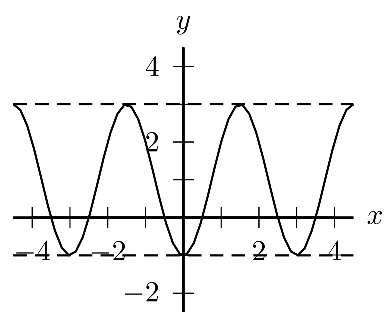
Solution: First, we vertically compress the graph of $h(x)$ by a factor of $\frac{1}{2}$. Then, we reflect the resulting graph across the x-axis.

Option A Option B $\boxed{\text{Option C}}$ NONE OF THESE

b. [3 points] Which, if any, of the figures below shows part of the graph of $y = h(2x + 2)$? *Note that the scale is smaller than in the original graph above. Be sure to pay attention to the scale indicated on the axes.*


(Option A)


(Option B)


(Option C)

Circle your one final answer below. (Only the answer you circle below will be graded.)

Solution: Note that $h(2x + 2) = h(2(x + 1))$. So the graph of $h(2x + 2)$ can be obtained from that of $h(x)$ by first compressing horizontally by a factor of $\frac{1}{2}$ and then shifting the resulting graph 1 unit to the left.

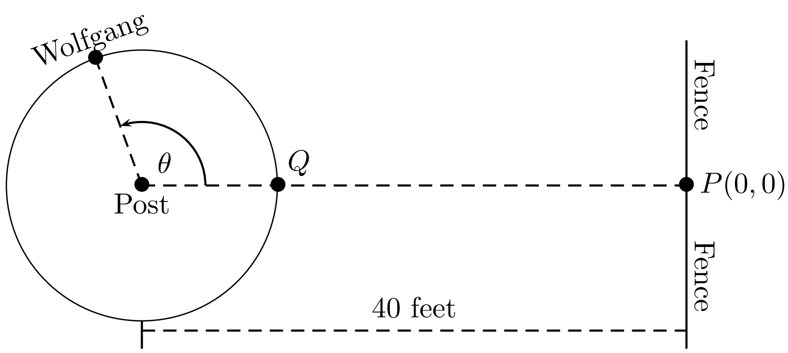
$\boxed{\text{Option A}}$ Option B Option C NONE OF THESE

10. [4 points] Suppose that the number of acorns in Squishy squirrel's nest is proportional to the cube of the number of squirrels currently living there. If there are 113 acorns in his nest when there are two squirrels living there, how many acorns will there be in Squishy's nest when there are four squirrels living there? *Remember to show your work carefully.*

Solution: Let A be the number of acorns in Squishy squirrel's nest. Let Q be the number of squirrels living there. Then we know that $A = kQ^3$ where k is some constant. If there are 113 acorns in his nest when there are two squirrels living there, then $113 = k(2^3)$ so $k = \frac{113}{8}$ and a general formula is $A = \frac{113}{8}Q^3$.
Therefore, when there are four squirrels living in the nest, $A = \frac{113}{8}(4^3) = \frac{113}{8}(64) = 113(8) = 904$. So there are 904 acorns in Squishy squirrel's nest when there are four squirrels living there.

Answer: 904 **acorns**

11. [7 points] Wolfgang the wolf is on a 10-foot long leash that is tied to a post that is 40 feet west of a fence.



Because he dislikes being on his leash, he stays 10 feet away from the post at all times.

a. [4 points] Suppose we think of the origin at the point P as shown in the diagram and that the unit of measurement is feet so that the coordinates of the post are $(-40, 0)$. Find Wolfgang's coordinates when he is at the angle θ shown in the diagram.
(*Your answer should be in terms of* θ.)

Solution: If the post were at the origin, his coordinates would be $(10\cos(\theta), 10\sin(\theta))$. Since the post is 40 feet to the left of the origin, his first coordinate will be 40 units less, i.e. $-40+10\cos(\theta)$. This does not change the second coordinate.

Answer: Wolfgang's coordinates are $\left(-40 + 10\cos(\theta),\ 10\sin(\theta)\right)$.

b. [3 points] Wolfgang starts walking counterclockwise from the point Q. The angle θ through which Wolfgang has walked is a function of the amount of time he has been walking. Let $\theta = z(t)$ be the angle (in radians) through which Wolfgang has walked after he has been walking for t minutes. Let $A(t)$ be the distance Wolfgang has traveled along the circle in t minutes. Find a function $f(t)$ such that $A(t) = f(z(t))$.

Solution: Using the arclength formula, $A(t) = 10\theta = 10z(t)$. We want to find a function $f(t)$ such that

$$f(z(t)) = A(t) \quad \text{i.e. so that } f(z(t)) = 10z(t).$$

The above shows that if the function f is given an input of $z(t)$, the output is $10z(t)$. Thus, f should be a function that takes its input and multiplies it input by 10. So $f(t) = 10t$.

Answer: $f(t) =$ $10t$

12. [10 points] Consider the functions f, g, and h defined as follows:

$$f(x) = a + bx \qquad g(x) = cx^d \qquad h(x) = w(1+r)^x$$

for nonzero constants a, b, c, d, r, and w with $r > -1$.

For each of the questions below, circle all the correct answers from among the choices provided, or circle NONE OF THESE if appropriate.

a. [2 points] The graph of which function(s) definitely has at least one horizontal intercept?

$\boxed{f(x)}$ $\qquad g(x) \qquad h(x) \qquad$ NONE OF THESE

Solution: $f(x)$ will always have a horizontal intercept since it is linear with nonzero slope.
If $d < 0$, e.g. if $g(x) = 10x^{-1}$, then $g(x)$ does not have a horizontal intercept.
$h(x)$ does not have a horizontal intercept because it is an exponential function and w is nonzero.

b. [2 points] The graph of which function(s) definitely has at least one horizontal asymptote?

$f(x) \qquad g(x) \qquad \boxed{h(x)}$ $\qquad$ NONE OF THESE

Solution: $f(x)$ is a linear function with nonzero slope so does not have a horizontal asymptote.
If $d > 0$, e.g. if $g(x) = 10x^2$, then it does not have a horizontal asymptote.
$y = h(x)$ is an exponential function so has horizontal asymptote $y = 0$.

c. [2 points] Which function(s) is(are) definitely invertible?

$\boxed{f(x)} \qquad g(x) \qquad \boxed{h(x)}$ $\qquad$ NONE OF THESE

Solution: The linear function with nonzero slope, $f(x)$, and the exponential function, $h(x)$ are definitely invertible. (They pass the horizontal line test, for example.)
$g(x)$ may or may not be invertible. For example, if $g(x) = 10x^2$, then it is not invertible.

d. [2 points] How many times could the graph of $f(x)$ intersect the graph of $h(x)$?

$\boxed{0} \qquad \boxed{1} \qquad \boxed{2} \qquad 3 \qquad 4$ $\qquad$ more than 4

Solution: A linear function can intersect an exponential function either 0, 1, or 2 times. For example, $y = x + 1$ and $y = 2(1+3)^x$ do not intersect, whereas $y = -x + 1$ and $y = 2(1+3)^x$ intersect exactly once, and $y = x + 5$ and $y = 2(1+3)^x$ intersect exactly two times. (They cannot intersect more than two times due to their long-run behavior.)

e. [2 points] Suppose the graph of h is concave up. Which of the following is(are) definitely true?

$\boxed{w > 0} \qquad w < 0 \qquad r > 0 \qquad r < 0$ $\qquad$ NONE OF THESE

Solution: An exponential function is concave up if and only if its initial value is positive.
That is, whether $-1 < r < 0$ or $r > 0$, if $w > 0$, then the graph of h will be concave up whereas if $w < 0$, the graph of h will be concave down.