# Math 105 — First Midterm

October 6, 2014

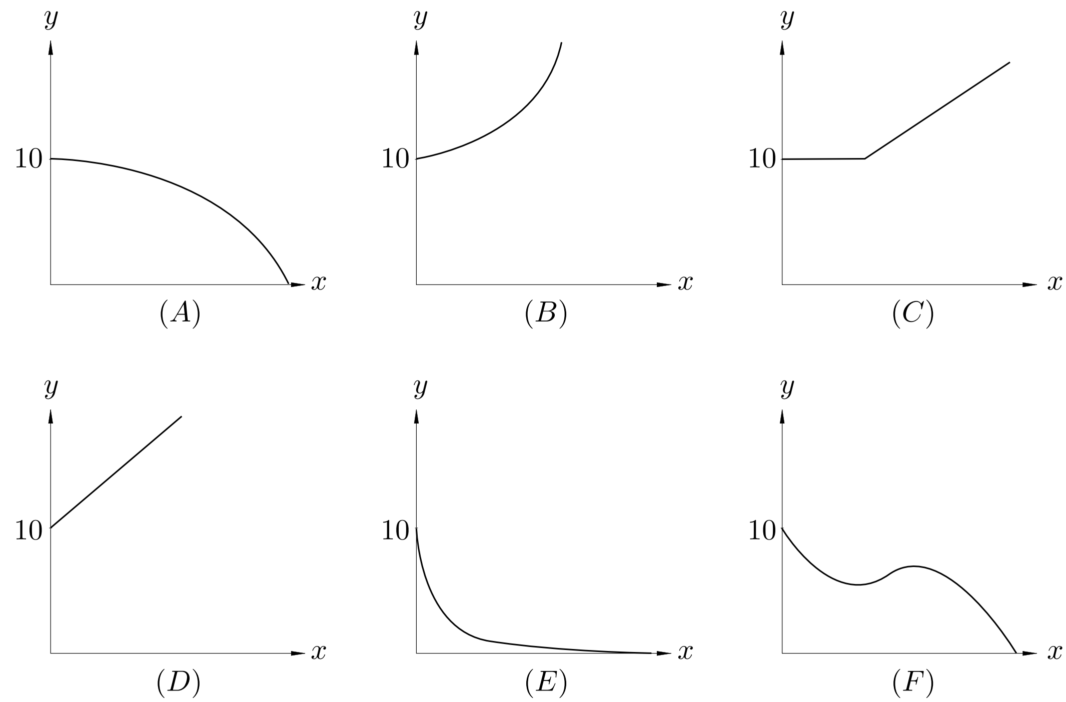
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 14 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

| Problem | Points | Score |
|---|---|---|
| 1 | 12 | |
| 2 | 10 | |
| 3 | 9 | |
| 4 | 16 | |
| 5 | 10 | |
| 6 | 5 | |
| 7 | 12 | |
| 8 | 14 | |
| 9 | 12 | |
| Total | 100 | |

**1**. [12 points] Use the graphs to answer the questions below.




**a**. [2 points] Which of the graphs above are concave down?

Answer=________________________.

*Solution:* A

**b**. [2 points] Which of the graphs above have constant rate of change?

Answer=________________________.

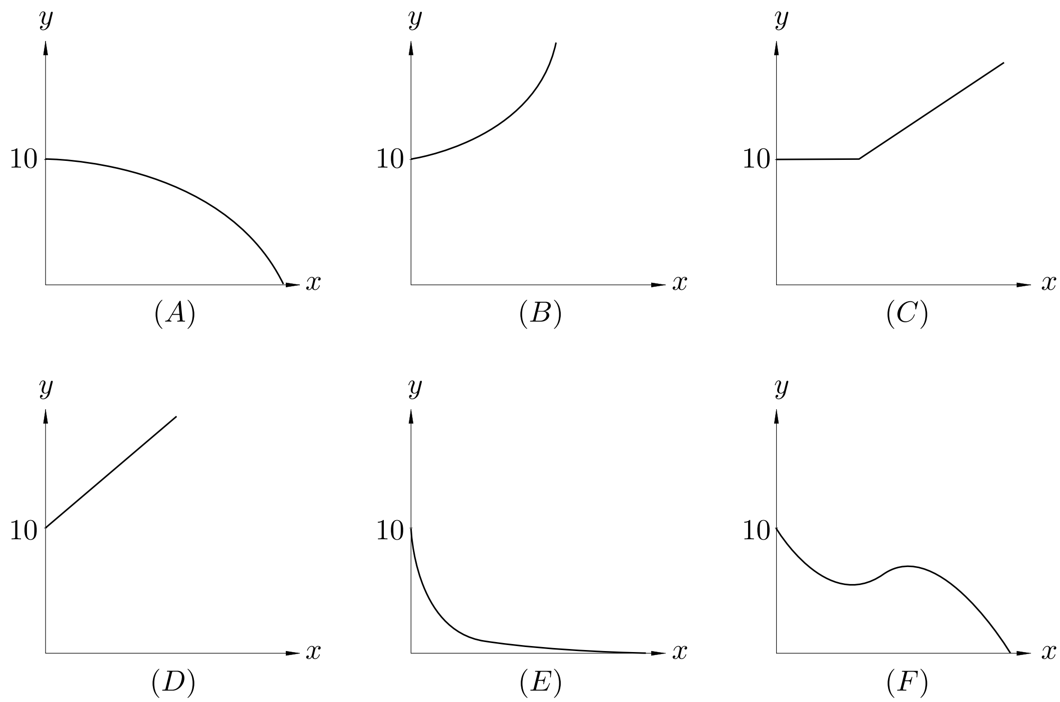
*Solution:* D

**c**. [2 points] Which of the graphs above are decreasing?

Answer=________________________.

*Solution:* A, E

*The graphs from the previous page have been copied here for your convenience*




Match each verbal description of a function below to its graph above.

**d**. [2 points] Let $f(x)$ be the amount of money in your savings account (in thousands of dollars) $x$ years after you make an initial deposit of \$10,000, assuming that the bank pays an annual interest rate of 2% and you do not withdraw or add any money to the account.

Answer=______________________.

*Solution:* B

**e**. [2 points] You place a mixing bowl weighing 10 grams on a weighing scale. Let $f(x)$ be the reading on your weighing scale (in grams) after adding $x$ grams of flour to the mixing bowl.

Answer=______________________.

*Solution:* D

**f**. [2 points] A rock is dropped from 10 meters above the ground. Let $f(x)$ be the height of the ball above the ground (in meters) $x$ seconds after you drop it.

Answer=______________________.

*Solution:* A

**2**. [10 points] Indicate if each of the following statements are true or false by circling the correct answer. No justification is required.

**a**. [2 points] Let $g$ be the inverse of the function $f$. If $a$ and $b$ are constants such that $a = f(b)$, then $b = g(a)$.

$\boxed{\text{True}}$ False

**b**. [2 points] The line $2x - 3y + 100 = 0$ is perpendicular to the line $12y + 18x = 1$.

$\boxed{\text{True}}$ False

*Solution:* The line $2x - 3y + 100 = 0$ has slope $m_1 = \frac{2}{3}$. The line $12y + 18x = 1$ has slope $m_2 = -\frac{3}{2}$. Since $m_1 m_2 = -1$, then the lines are perpendicular.

**c**. [2 points] Some of the values of the function $K$ are given in the table.

| $u$ | $-3$ | $-1$ | $2$ |
|---|---|---|---|
| $K(u)$ | 2 | 3 | 4 |

The function $K$ could be linear.

True $\boxed{\text{False}}$

*Solution:* Looking at the rate of change between consecutive points in the table:

$$m_1 = \frac{3-2}{-1+3} = \frac{1}{2} \quad m_2 = \frac{4-3}{2+1} = \frac{1}{3} \quad \text{then } K(u) \text{ can't be a linear function.}$$

**d**. [2 points] Some of the values of the function $Q$ are given in the table.

| $z$ | $-3$ | $-1$ | $1$ | $3$ |
|---|---|---|---|---|
| $Q(z)$ | 5 | 0.5 | $-2$ | $-4$ |

The graph of the function $Q$ could be concave up .

$\boxed{\text{True}}$ False

Looking at the rate of change between consecutive points in the table:

$$m_1 = \frac{0.5-5}{-1+3} = -2.25, \quad m_2 = \frac{-2-0.5}{1+1} = -1.25, \quad m_3 = \frac{-4+2}{3-1} = -1.$$

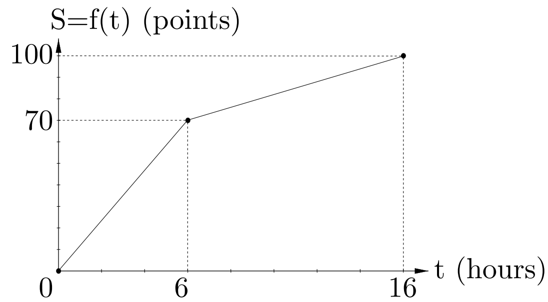
then the graph of the function $Q(x)$ can be concave up.

**e**. [2 points] If $f(x) = 2x + 1$ and $g(x) = x^2 + 1$ then $f(g(x)) = 2x^2 + 3$.

$\boxed{\text{True}}$ False

*Solution:* $f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) + 1 = 2x^2 + 3$.

**3**. [9 points] Let $t$ be the number of hours you spent studying for a midterm, which is worth 100 points. Let $S$ be your score in the midterm, and let $G$ be the letter grade you get. The graph of the function $f$ so that $S = f(t)$ is drawn below.




Also, the function $h$ so that $G = h(S)$ is given by the table below.

| $S$ | 0 – 35 | 36 – 50 | 51 – 74 | 75 – 86 | 87 – 100 |
|---|---|---|---|---|---|
| $G = h(S)$ | E | D | C | B | A |

For example, if you get a score between 0 and 35, you get an E grade.

**a**. [5 points] Find a formula for the function $f$ written as a piecewise defined function.

*Solution:*
The slope of the line between $(0, 0)$ and $(6, 70)$ is $\frac{70}{6}$. Thus, when $0 \le t \le 6$, $f(t) = \frac{70}{6}t$. Also, the slope of the line between $(6, 70)$ and $(16, 100)$ is $\frac{100-70}{16-6} = 3$, so the equation of this line is of the form $S = 3t + c$. Plug in the point $(6, 70)$ to solve for $c = 52$. Thus, when $6 \le t \le 16$, $f(t) = 3t + 52$. Putting this together, we obtain

$$f(t) = \begin{cases} \frac{70}{6}t & \text{if } 0 \le t \le 6 \\ 3t + 52 & \text{if } 6 \le t \le 16 \end{cases}$$

**b**. [2 points] What is the minimum amount of time you need to spend studying to get an $A$ in the midterm? Include units and your answer must be exact or accurate up to 2 decimal places.

*Solution:* The minimum score needed to get an A is 87. Using the equation for $f$, solve $3t + 52 = 87$, to obtain $t = \frac{35}{3} = 11.66$ hours.
Answer: 11.66 hours.

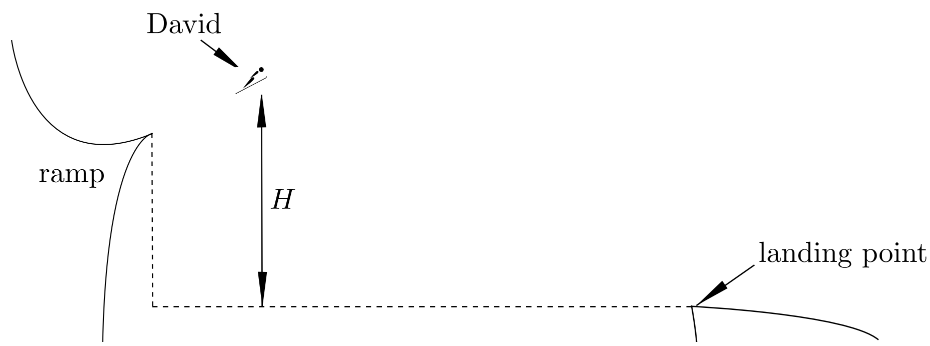
**c**. [2 points] Give a practical interpretation of the statement $h(f(6)) = \text{C}$. Use complete sentences in your answer.

*Solution:* After studying for 6 hours, I will get the letter grade C for the midterm.

**4.** [16 points] David is a professional extreme athlete. In one of his stunts, he jumps off a ski ramp. David's height $H$ (in m) above his landing point, from the moment he leaves the ramp until he lands, is given by the function

$$H = f(t) = -5t^2 + 8t + 15.$$

In this formula, $t$ is the time (in seconds) after David leaves the ramp.




**a.** [3 points] Find the exact time it took David to travel from the ramp to his landing point? Include units.

*Solution:* The time it takes for David to reach his landing point has to satisfy $-5t^2 + 8t + 15 = 0$. Using the quadratic formula, we have

$$t = \frac{-8 \pm \sqrt{8^2 + 300}}{-10} = \frac{8 \pm \sqrt{364}}{10} = 0.8 \pm \sqrt{3.64}.$$

Thus, $t = 0.8 + \sqrt{3.64}$ seconds.

**b.** [5 points] Use the method of completing the square to write the formula for $f(t)$ in vertex form. Carefully show your algebraic work step-by step.

*Solution:*

$$\begin{aligned} f(t) = -5t^2 + 8t + 15 &= -5(t^2 - \frac{8}{5}t) + 15 \\ &= -5\left(t^2 - \frac{8}{5}t + \frac{16}{25} - \frac{16}{25}\right) + 15 \\ &= -5\left(t^2 - \frac{8}{5}t + \frac{16}{25}\right) + \frac{16}{5} + 15 \\ &= -5\left(t - \frac{4}{5}\right)^2 + \frac{91}{5} \end{aligned}$$

*We rewrote the problem in this page for your convenience*

David is a professional extreme athlete. In one of his stunts, he ski jumps off a ski ramp. David's height $H$ (in m) above his landing point, from the moment he leaves the ramp until he lands, is given by the function

$$H = f(t) = -5t^2 + 8t + 15.$$

In this formula, $t$ is the time (in seconds) after David leaves the ramp.

**c**. [2 points] What is the exact value of David's maximum height above his landing point during his jump? Include units.

*Solution:* Since the vertex of the quadratic is at $\left(\frac{4}{5}, \frac{91}{5}\right)$, then the maximum height is at $\frac{91}{5}$ meters.
Answer=$\frac{91}{5}$ or 18.2 meters

**d**. [2 points] How high is the ramp above his landing point? Include units.

*Solution:* $f(0) = 15$ meters.
Answer:__________________

**e**. [4 points] What is the domain and range of $H = f(t)$ in the context of this problem? Express your answer using inequalities or interval notation. Your answer has to be exact.

*Solution:* Domain:$[0, 0.8 + \sqrt{3.64}]$, Range:$[0, \frac{91}{5}]$

**5**. [10 points] While ski jumping, David broke his leg and was taken to the hospital. The hospital doctor administered a painkiller to David at noon. At 3 pm, the concentration of the painkiller in David's blood was 10 mg per liter and at 5 pm, it fell to 6 mg per liter. Let $C(t)$ be the concentration (in mg per liter) of the painkiller in David's blood $t$ hours after noon. Suppose that the function $C$ is decreasing exponentially.

**a**. [6 points] Find a formula for $C(t)$. Show all your work. Your answer must be exact.

*Solution:* Since $C$ is an exponential function, $C(t) = ab^t$. Plug in $(3, 10)$ and $(5, 6)$ to obtain

$$10 = ab^3 \quad \text{and} \quad 6 = ab^5.$$

By taking the ratio, we get $b^2 = \frac{3}{5}$, so $b = \sqrt{0.6}$. Thus, $a = \frac{10}{\sqrt{0.6}^3}$, and so
$C(t) = \frac{10}{\sqrt{0.6}^3}\left(\sqrt{0.6}\right)^t$.

**b**. [4 points] What is the hourly percentage growth rate of $C(t)$ and the initial concentration of painkiller in David's blood? Include units when appropriate. Your answer must be exact or accurate up to one decimal place.

*Solution:* Hourly percentage growth rate= $-22.5\%$. Initial concentration= 21.5 mg/liter.

**6**. [5 points] For each of the following functions, write down its growth factor if the function is exponential or NONE if the function is not exponential.

(i) $f(t) = 2t^3$ Answer=________________

*Solution:* NONE

(ii) $g(t) = 2^t 3^t$ Answer=________________

*Solution:* 6

(iii) $h(t) = (3^{-t})^2$ Answer=________________

*Solution:* $\frac{1}{9}$

**7**. [12 points] Include all your work in the following problems to receive full credit.

**a**. [6 points] At the supermarket, you decide to buy blueberries and mangos. The price of blueberries is \$5.75 per pound and mangos cost \$3.20 per pound. Suppose that you spend \$30 buying $B$ pounds of blueberries and $M$ pounds of mangos. Let $f$ be the function such that $B = f(M)$.

(i) Find a formula for $f$.

*Solution:* $30 = 5.75B + 3.20M \Rightarrow B = f(M) = \frac{30-3.20M}{5.75}$

(ii) Find the vertical intercept of the graph of the function $f$, and interpret this intercept using complete sentences. Include units, and your answer must be exact or accurate up to 2 decimal places.

*Solution:* Vertical intercept= $f(0) = \frac{30}{5.75} = 5.22$ pounds.
Practical interpretation: The vertical intercept is the number of pounds of blueberries I can buy if I spend all \$30 buying blueberries.

**b**. [6 points] A supermarket opens everyday at 8 am and closes at 6 pm. The supermarket manager notices that the amount of clients during a day is given by a quadratic function. Let $C(t)$ be the amount of clients in the supermarket $t$ hours after the store opened. Find a formula for $C(t)$ if there are 250 clients in the store at 10 am, and there are no clients when the store opens and closes.

*Solution:* Since $C(0) = 0$ and $C(10) = 0$ and $C$ is a quadratic function, we have that its factored formula is

$$C(t) = a(t-10)t$$

for some $a$. Plug in $(t, C) = (2, 250)$ to obtain $250 = -16a$ and solve for $a = -\frac{125}{8}$. Thus, $C(t) = -\frac{125}{8}t(t-10)$.

*Solution:* Let $C(t) = at^2 + bt + c$. Since $C(0) = 0$, we have $C(t) = at^2 + bt$ for some $a, b$. Plug in $(10, 0)$ to obtain $0 = 100a + 10b$ and plug in $(2, 250)$ to obtain $250 = 4a + 2b$. Hence

$$0 = 100a + 10b \qquad\qquad 250 = 4a + 2b.$$

Solving for $a$ and $b$, you get $a = -\frac{125}{8}$ and $b = \frac{625}{4}$. Thus, $C(t) = -\frac{125}{8}t^2 + \frac{625}{4}t$.

**8**. [14 points] An ice cube is left to melt in a warm room. Let $V = f(t)$ be the volume of the ice cube (in $\text{cm}^3$) $t$ seconds after it starts melting. Also, as the ice cube melts, a circular puddle of water of radius $r$ (in cm) and area $A$ (in $\text{cm}^2$) starts forming around it. Let $g$ and $h$ be functions such that $r = g(t)$ and $A = h(r)$. You may assume $f$, $g$ and $h$ are invertible.

**a**. [6 points] Select a mathematical expression from the list below that represents each of the following statements.

(i) When the volume of the ice cube is 30 $\text{cm}^3$, the radius of the water puddle around the ice cube is 6 cm.

*Solution:* E

Answer: ______________________

(ii) The radius of the water puddle grows by 6 cm between 20 and 30 seconds after the ice cube started melting.

*Solution:* G

Answer: ______________________

(iii) Between 20 and 30 seconds after the ice cube started melting, the radius of the water puddle grows, on average, by 6 cm per second.

*Solution:* B

Answer: ______________________

**A)** $f(g(6)) = 30$ **B)** $\dfrac{g(30) - g(20)}{10} = 6$ **C)** $g(30) = 6$

**D)** $\dfrac{g(30) - g(20)}{20} = 6$ **E)** $f(g^{-1}(6)) = 30$ **F)** $\dfrac{g(30) + g(20)}{2} = 6$

**G)** $g(30) - g(20) = 6$ **H)** $f(6) = 30$ **J)** $g(20) - g(30) = 6$

**b**. [4 points] The following statements are practical interpretations of mathematical expressions (not necessarily the ones listed above). Write the mathematical expression in each case.

(i) The time (in seconds) it takes for the radius of the water puddle around the ice cube to be 7 cm.

*Solution:* $g^{-1}(7)$

Answer: ______________________

(ii) The area (in $\text{cm}^2$) of the circular water puddle formed around the ice cube 9 seconds after the ice cube started melting.

*Solution:* $h(g(9))$

Answer: ______________________

**c**. [4 points] Assume that the domains of $f$ and $g$ is the interval of time it takes for the entire ice cube to melt. Indicate if the following functions are increasing, decreasing or neither.

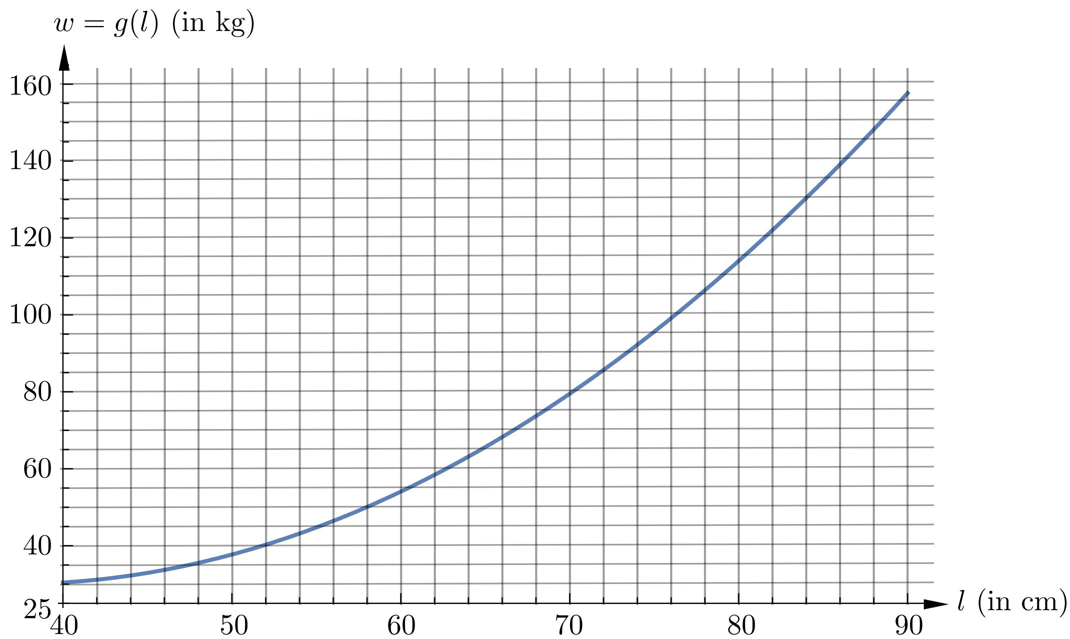
*Solution:* $f(t)$ represents the volume of the ice cube (in $\text{cm}^3$) at time $t$ (in seconds). Since the ice cube is melting, then $f(t)$ is decreasing. $h(g(t))$ represents the area of the water puddle forming around the ice cube (in $\text{cm}^2$) at time $t$ (in seconds). Since the ice cube is melting, the area of the water puddle is increasing.

$f(t)$ is decreasing and $h(g(t))$ is increasing.

**9**. [12 points] You would like to investigate the relationship between the swimming speed $S$ (in cm/sec), the weight $w$ (in kg) and the length $l$ (in cm) of salmon. Let $f$ and $g$ be invertible functions that take as input the length of the salmon and give as output its swimming speed and weight respectively. In other words, $S = f(l)$ and $w = g(l)$. You measured the swimming speed and the length of six salmons. The data you obtained is summarized in the table below.

| | Salmon 1 | Salmon 2 | Salmon 3 | Salmon 4 | Salmon 5 | Salmon 6 |
|---|---|---|---|---|---|---|
| $l$ | 60 | 80 | 50 | 85 | 76 | 40 |
| $S = f(l)$ | 148 | 161 | 140 | 163 | 158 | 130 |

The graph of $g$ is drawn below.




**a**. [6 points] Find the value of the following expressions. Include units.

$g^{-1}(100)$=____________ $f(80)$=____________ $f^{-1}(140)$=____________

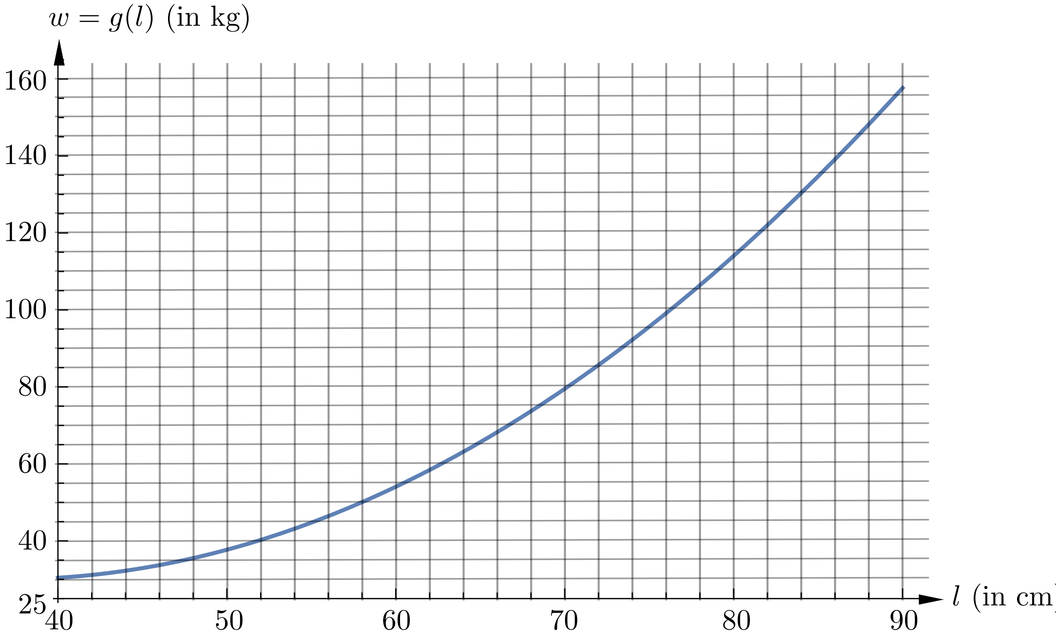
*Solution:* $g^{-1}(100) = 76$ cm, $f(80) = 161$ cm/s, $f^{-1}(140) = 50$ cm.

**b**. [2 points] What is the **weight** of a salmon that swims at a speed of 130 cm/sec?

*Solution:* A salmon that can swim at 130 cm/sec has length 40 cm. According to the graph, the weight of a salmon 40 cm long is 30 kg.

*The graph and table from the previous page has been copied here for your convenience*

| | Salmon 1 | Salmon 2 | Salmon 3 | Salmon 4 | Salmon 5 | Salmon 6 |
|---|---|---|---|---|---|---|
| $l$ | 60 | 80 | 50 | 85 | 76 | 40 |
| $S = f(l)$ | 148 | 161 | 140 | 163 | 158 | 130 |




**c**. [4 points] Find the average rate of change of the **weight** of a salmon as a function of its swimming speed over the interval between $S = 148$ and $S = 158$. Show all your work to receive full credit. Include units.

*Solution:* When the swimming speed of a salmon is 148 cm per second, its length is 60 cm. If the swimming speed of a salmon is 158 cm per second, its length is 76 cm. The weight of a salmon of 60 cm and 76 cm in length is 55 kg and 100 kg. Hence the average rate of change of the weight of a salmon over the interval between $S = 148$ and $S = 158$ is given by

$$\frac{g(f^{-1}(158)) - g(f^{-1}(148))}{158 - 148} = \frac{100 - 55}{158 - 148} = 4.5 \quad \frac{\text{kg}}{\text{cm/sec}}.$$