Math 105 — Second Midterm

November 10, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

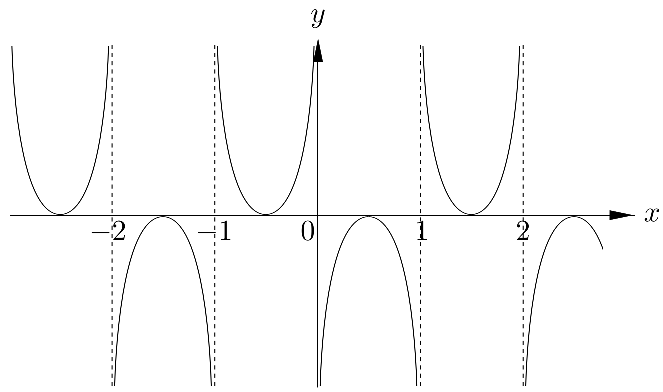
1. **Do not open this exam until you are told to do so.**
2. This exam has 13 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	12	
2	10	
3	15	
4	8	
5	9	
6	9	
7	8	
8	10	
9	11	
10	8	
Total	100	

1. [12 points] For each of the following graphs of functions, write down

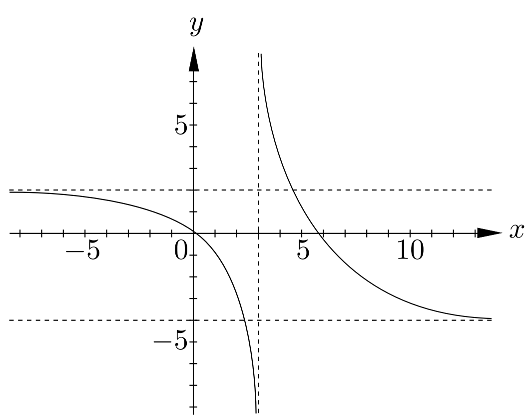
- the period if the function could be periodic with a period less than 10, or **NONE** otherwise.
- the equation of a horizontal asymptote if it has horizontal asymptotes, or **NONE** otherwise.
- the equation of a vertical asymptote if it has vertical asymptotes, or **NONE** otherwise.

a. [6 points]



Solution:
Period=2.
Vertical asymptote: $x = 1$.
Horizontal asymptote: None.

b. [6 points]



Solution:
Period=None.
Vertical asymptote: $x = 3$.
Horizontal asymptote: $y = 2$ or $y = -4$.

2. [10 points] Indicate if the following statements are true or false. No explanation is required.

a. [2 points] The period of the function $f(x) = \tan(\frac{x}{k})$ is $k\pi$ for any positive number k.

[True] False

b. [2 points] If the function $f(x)$ is periodic with period 2, then $f(x) = f(x + 4)$.

[True] False

c. [2 points] The amplitude of the function $f(x) = 1 + \sin(x)$ is equal to two.

True [False]

d. [2 points] The function $f(x) = \dfrac{x}{1 + x^2}$ is an even function.

True [False]

e. [2 points] The mass of the carbon isotope Carbon-14 decays exponentially with time. If it takes 5730 years for the quantity of Carbon-14 to decay to half of its initial amount, then it takes a further 5730 years for the quantity of Carbon-14 to decay to a quarter of its initial amount.

[True] False

3. [15 points] At a restaurant, the temperature of a bowl of soup S (in °F) t minutes after it is served is given by the function $S = f(t) = 50 + 130e^{-0.07t}$.

a. [2 points] At what temperature was the soup served? Include units.

Solution: $f(0) = 180$°F.

b. [4 points] How long does it take for the temperature of soup to reach 90°F? Find your answer algebraically. Show all your work. Your answer must be exact or accurate up to two decimal places.

Solution: When the temperature of the soup is 90°F, t satisfies the equation

$$\begin{aligned} 90 &= 50 + 130e^{-0.07t} \\ \frac{4}{13} &= e^{-0.07t} \\ \ln\left(\frac{4}{13}\right) &= \ln(e^{-0.07t}) \\ \ln\left(\frac{4}{13}\right) &= -0.07t. \\ t &= -\frac{\ln(\frac{4}{13})}{0.07} \approx 16.84 \text{ minutes.} \end{aligned}$$

c. [4 points] Does the function f have vertical or horizontal asymptotes? If it does, write down their equations, otherwise write **None**.

Solution:
Equation of the horizontal asymptote: $S = 50$.

Equation of the vertical asymptote: NONE

Problem continues on the next page

The statement of the problem is included here for your convenience.

At a restaurant, the temperature of a bowl of soup S (in °F) t minutes after it is served is given by the function $S = f(t) = 50 + 130e^{-0.07t}$.

d. [5 points] At the same restaurant, the temperature of a cup of coffee C (in °F) t minutes after it is served is given by the function $C = g(t) = 50 + 150\,(0.85)^t$. A bowl of soup and a cup of coffee are served at the same time. How long does it take for the temperature of the soup and the coffee to be equal after they are served? Show all your work. Your answer must be exact or accurate up to two decimal places.

Solution: When the temperature of the coffee and the soup are equal, we have

$$\begin{aligned} 50 + 130e^{-0.07t} &= 50 + 150(0.85)^t \\ 130e^{-0.07t} &= 150(0.85)^t \\ \frac{e^{-0.07t}}{(0.85)^t} &= \frac{15}{13}. \\ \left(\frac{e^{-0.07}}{0.85}\right)^t &= \frac{15}{13}. \\ t\ln\left(\frac{e^{-0.07}}{0.85}\right) &= \ln\left(\frac{15}{13}\right) \\ t &= \frac{\ln\left(\frac{15}{13}\right)}{\ln\left(\frac{e^{-0.07}}{0.85}\right)} \approx 1.55 \text{ minutes .} \end{aligned}$$

Or

$$\begin{aligned} 50 + 130e^{-0.07t} &= 50 + 150(0.85)^t \\ 130e^{-0.07t} &= 150(0.85)^t \\ \ln(130e^{-0.07t}) &= \ln(150(0,85)^t. \\ \ln(130) - 0.07t &= \ln(150) + t\ln(0.85). \\ t(-\ln(0.85) - 0.07) &= \ln(150) - \ln(130). \\ t &= \frac{\ln(150) - \ln(130)}{-\ln(0.85) - 0.07} \approx 1.55 \text{ minutes .} \end{aligned}$$

4. [8 points] Let $h(t)$ be the height (in meters) of a roller coaster carriage above the ground during a roller coaster ride, t seconds after the ride starts.

a. [2 points] Let $g(t)$ be the height (in **feet**) of the roller coaster carriage above the ground during a roller coaster ride, t seconds after the ride starts. Write a formula for the function g in terms of the function h and the variable t. (1 meter= 3.28 feet)

Solution: $g(t) = 3.28h(t)$.

b. [3 points] The roller coaster carriage reaches its highest point p seconds after the ride starts. Write an equation that expresses the following fact:
*Three seconds after the the roller coaster carriage reaches its highest point, its height (***in meters***) above the ground is twice its height at the starting point.*

Solution: $h(p+3) = 2h(0)$.

c. [3 points] During a renovation of the amusement park, the following modifications are made to the roller coaster, in the stated order.

1. The starting point of the roller coaster is moved to where the roller coaster carriage would have been 30 seconds into the ride from the old starting point.
2. The entire roller coaster is raised onto a 2 meter high platform.

After the renovation, let $k(t)$ be the height (in **meters**) of the roller coaster carriage above the ground during a roller coaster ride, t seconds after the ride starts. Write a formula for the function k in terms of the function h and the variable t.

Solution: $k(t) = h(t+30) + 2$.

5. [9 points] As part of an experiment, a new pesticide was used in an apartment building to reduce the population of termites. Let $P(t) = 6e^{-0.8t}$ be the termite population (in thousands) t days after the pesticide was applied.

a. [1 point] What is the continuous growth rate per day of the population of termites given by the function $P(t)$?

Solution: Continuous growth rate$=-0.8$ or -80%.

b. [2 points] Let $G(T)$ be the population of termites (in thousands) T **weeks** after the pesticide was applied. Write down a formula for $G(T)$ only in terms of T.

Solution: $G(T) = P(7T) = 6e^{-5.6T}$.

c. [2 points] Find the growth factor of the function $G(T)$. Your answer must be exact.

Solution: Growth factor of $G(T) = e^{-5.6}$.

d. [4 points] If the concentration of the pesticide is increased, then the termite population (in thousands) t days after the pesticide was applied is given by the function $L(t) = 6e^{-kt}$, where k is a positive number depending on the concentration of the pesticide. The higher the concentration of the pesticide, the more quickly the termite population will decrease.

Indicate if the following statements are True or False. Circle your answer. No explanation is required.

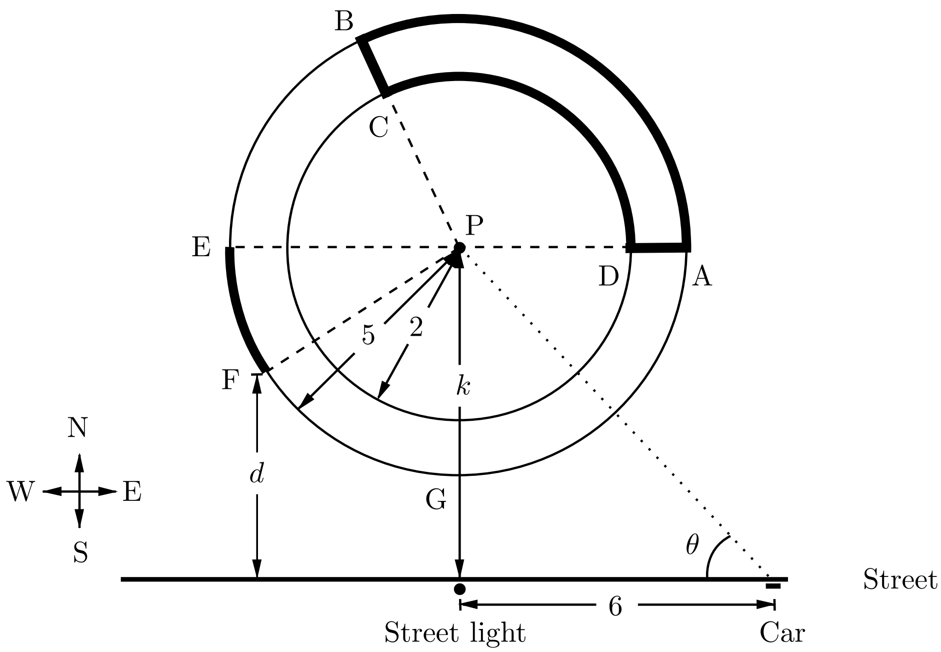
i) The higher the concentration of the pesticide, the bigger the number k will be.

[True] False

ii) The higher the concentration of the pesticide, the smaller the value of $\lim_{t\to\infty} L(t)$ will be.

True [False]

6. [9 points] At a park, there are two circular tracks that are centered at a common flagpole (at point P). The two tracks have radii 2 and 5 km respectively (see the figure below). A street that runs in the east-west direction is located k kilometers south of the flagpole.



a. [4 points] Albert decides to run on the tracks starting at the point A on the east end of the longer track . He runs along the longer track counterclockwise until he reaches point B. Then he runs from point B towards the flagpole until point C on the shorter track. He continues clockwise along the shorter track until point D. From there, he runs east to point A (see the bolded path in the figure). If the distance Albert ran along the longer track between the points A and B is 7 km, what is the total distance he ran?

Solution: Since Albert ran 7 km along the longer track between A and B with radius 5 km, then the arc length formula $s = r\theta$ yields angle $BPA = \frac{s}{r} = \frac{7}{5}$ radians. This angle can be used to find the length of the arc CD in the shorter track with radius 2

$$\text{length of arc } CD = r(\text{angle } BPA) = 2\left(\frac{7}{5}\right) = \frac{14}{5} \text{ kilometers.}$$

Since the distance between B and C and the distance between D and A are both $5-2=3$ km, the total distance ran by Albert is $3+3+7+\frac{14}{5} = 15.8$ km.

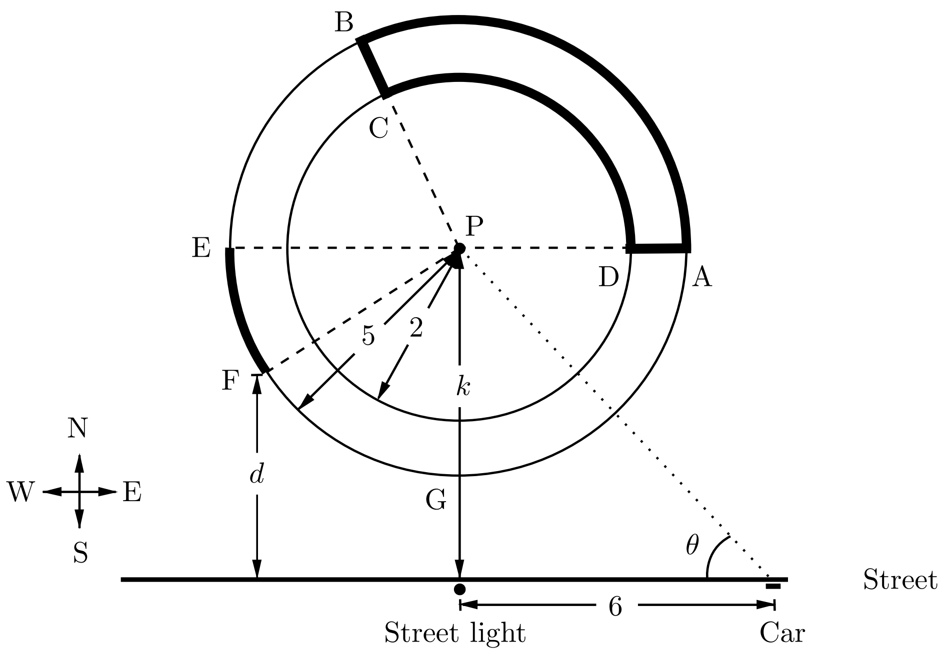
b. [3 points] John starts running at point E, which is the furthest point directly west of the flagpole on the longer track. He plans to run on the track in the counterclockwise direction to the point G, which is directly south of the flagpole. He stops at point F which is a third of the way between point E and G on the track. What is John's distance d (in kilometers) to the street at this point? Your answer may depend on k.

Solution: Since F is a third of the way from E to G, the angle EPF is $\frac{\pi}{6}$. This implies that the distance from F to the line EP is $5\sin(\frac{\pi}{6}) = \frac{5}{2}$. Thus, $d = k - \frac{5}{2}$ km.

Problem continues on the next page

The statement of the problem is included here for your convenience.

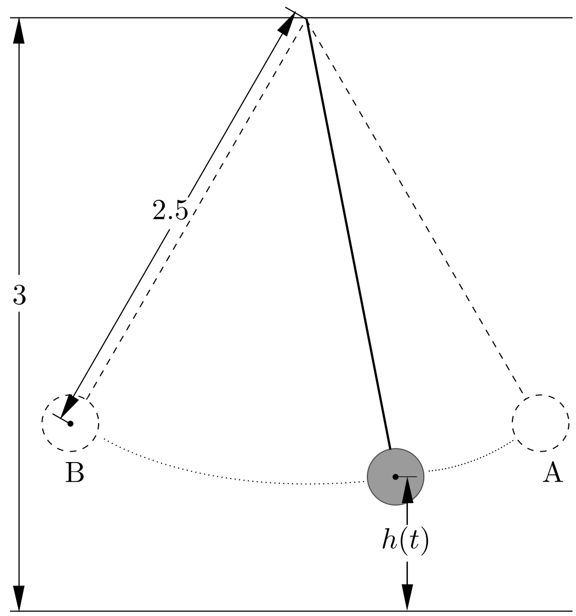
At a park, there are two circular tracks that are centered at a common flagpole (at point P). The two tracks have radii 2 and 5 km respectively (see the figure below). A street that runs in the east-west direction is located k kilometers south of the flagpole.



c. [2 points] Directly south of the flagpole, there is a street light on the street. A car is parked 6 km from the streetlight along the street, and the line connecting the car with the flagpole makes an angle θ with the street (see the figure). Find a formula for the distance k (in kilometers) between the flagpole and the street light in terms of θ.

Solution: Consider the right-angled triangle whose vertices are the car, the street light and P. In that triangle, $\tan(\theta) = \frac{k}{6}$, so $k = 6\tan(\theta)$ km.

7. [8 points] The pendulum drawn below is a sphere that is hung from the ceiling by a piece of string that is 2.5 meters long. The ceiling is 3 meters above the floor, and the pendulum is swinging in between the points A and B as shown in the picture below.



Let $H = h(t)$ be the distance (in meters) between the center of the pendulum and the ground at time t (in seconds). Suppose that the function h is periodic, and that the midline of h is the line $H = 1$.

a. [2 points] If it takes two seconds for the pendulum to move from A to B (and also from to B to A), what is the period of the function h?

Solution: Period of h= 2 seconds.

b. [2 points] What is the minimum value of the function h?

Solution: Minimum value of $h = 3 - 2.5 = 0.5$ m.

c. [2 points] What is the amplitude of h?

Solution: Amplitude of $h = 1 - 0.5 = 0.5$ m.

d. [2 points] What is the maximum value of the function h?

Solution: Maximum value of $h = 1 + 0.5 = 1.5$ m.

8. [10 points] The Richter scale is a function r that takes as input the amount of energy E (in kJ) released in an earthquake, and outputs a number. The function r can be given by the formula

$$r(E) = \frac{2}{3}\log\left(\frac{E}{E_0}\right).$$

a. [5 points] An earthquake that releases 63,000 kJ of energy is assigned the number 2 by the Richter scale. What is the value of E_0? Find your answer algebraically. Show all your work.

Solution: Since an earthquake that releases 63,000 kJ of energy is assigned the number 2, we have

$$2 = \frac{2}{3}\log\left(\frac{63,000}{E_0}\right)$$
$$3 = \log\left(\frac{63,000}{E_0}\right).$$
$$10^3 = 10^{\log\left(\frac{63,000}{E_0}\right)}$$
$$1000 = \frac{63,000}{E_0}$$
$$E_0 = 63.$$

b. [5 points] Let E_A and E_B be the energy (in kJ) released during Earthquake A and Earthquake B respectively. Suppose that the amount of energy released during Earthquake A was 1000 times the amount of energy released during Earthquake B. What is $r(E_A) - r(E_B)$? Simplify as much as possible. Your answer should not involve any of the constants E_A or E_B.

Solution:

$$\begin{aligned} r(E_A) - r(E_B) &= \frac{2}{3}\log\left(\frac{E_A}{E_0}\right) - \frac{2}{3}\log\left(\frac{E_B}{E_0}\right) \\ &= \frac{2}{3}\log\left(\frac{E_A}{E_B}\right) \\ &= \frac{2}{3}\log\left(\frac{1000E_B}{E_B}\right) \\ &= \frac{2}{3}\log(1000) = \frac{2}{3}(3) = 2. \end{aligned}$$

9. [11 points] Solve the following equations algebraically. Show all your work. Your answers should be **exact**.

a. [4 points] $\log(x+1) - \log(x) = 1$.

Solution:

$$\begin{aligned}
&\log(x+1) - \log(x) = 1 \\
&\log\left(\frac{x+1}{x}\right) = 1 \\
&\frac{x+1}{x} = 10 \\
&x + 1 = 10x \\
&x = \frac{1}{9}
\end{aligned}$$

b. [3 points] $e^{3\ln(q)} = 2q^3 - 5$.

Solution:

$$\begin{aligned}
&e^{3\ln(q)} = 2q^3 - 5 \\
&e^{\ln(q^3)} = 2q^3 - 5 \\
&q^3 = 2q^3 - 5 \\
&q^3 = 5 \\
&q = 5^{\frac{1}{3}}
\end{aligned}$$

c. [4 points] $10\log(z^2) = \log(z) + 1$.

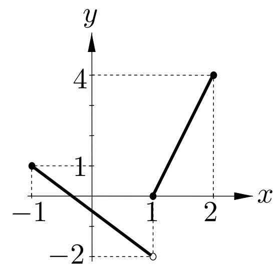
Solution:

$$\begin{aligned}
10\log(z^2) &= \log(z) + 1 \\
20\log(z) &= \log(z) + 1 \\
19\log(z) &= 1 \\
\log(z) &= \frac{1}{19} \\
z &= 10^{\frac{1}{19}}
\end{aligned}$$

or

$$\begin{aligned}
&\log(z^{20}) = \log(z) + 1 \\
&\log(z^{20}) - \log(z) = 1 \\
&\log(z^{19}) = 1 \\
&z^{19} = 10 \\
&z = 10^{\frac{1}{19}}.
\end{aligned}$$

10. [8 points] The graph of a function $f(x)$ is drawn below.



Match each of the following functions to their graphs below by circling one of the two available options. No justification is required.

a. [2 points] $y = \frac{1}{2}f(x+1) - 1$. **A** **C**

b. [2 points] $y = f(2x-1) + 1$. **B** **E**

c. [2 points] $y = -2f(x+1) + 1$. **C** **D**

d. [2 points] $y = f\left(-\frac{1}{2}(x-1)\right) + 1$. **E** **F**