Math 105 — Final Exam

December 12, 2014

Name: EXAM SOLUTIONS

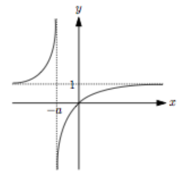
Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	7	
3	6	
4	8	
5	8	
6	6	
7	9	
8	9	
9	10	
10	10	
11	15	
12	6	
Total	100	

1. [6 points] Circle a possible equation for the following graphs. Here, a is a positive constant.

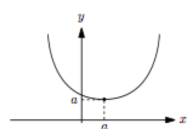
a. [2 points]



Solution:

$$f(x) = \frac{x+1}{x+a} \qquad f(x) = \frac{x}{x-a} \qquad \boxed{f(x) = \frac{x}{x+a}} \qquad f(x) = \frac{x+1}{x-a}$$

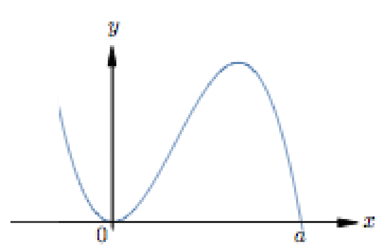
b. [2 points]



Solution:

$$\boxed{f(x) = (x-a)^4 + a} \qquad f(x) = x^2 + a \qquad f(x) = (x-a)^3 + a \qquad f(x) = (x+a)^2 + a$$

c. [2 points]



Solution:

$$f(x) = (x-a)x^2 \qquad \boxed{f(x) = -(x-a)x^2} \qquad f(x) = -(x-a)x \qquad f(x) = -(x-a)^2x$$

2. [7 points]

a. [3 points] Let $f(x) = \ln(x)$ and let g be the function whose graph is obtained by performing the following transformations to the graph of f, in the following order:

1) A horizontal stretch by a factor of 3.
2) A horizontal shift to the left by 1.
3) A vertical compression by factor of $\frac{1}{5}$.

Write down a formula for $g(x)$

Solution: $g(x) = \frac{1}{5}\ln\left(\frac{1}{3}x(x+1)\right)$

b. [4 points] The graph $y = K(x)$ has the line $y = 2$ as its horizontal asymptote and a horizontal intercept at $(1, 0)$. Let H be the function given by the formula $H(x) = -\frac{1}{7}K(2x+3)$. Find the the horizontal intercept and the equation of the horizontal asymptote of the graph $y = H(x)$.

Solution: Horizontal asymptote: $y = -\dfrac{2}{7}$.

Horizontal intercept: $(-1, 0)$

3. [6 points]

a. [4 points] Let a be a non-zero number. Find the zeroes of the polynomial $3x(x^2 + ax)^2$ and indicate if each zero is a double zero or a triple zero.

Solution: $p(x) = 3x^3(x + a)^2$. Zeros : $x = 0$ (triple zero) and $x = -a$ (double zero).

b. [2 points] Let f and g be functions given by the formulas

$$f(x) = \sqrt{1 + 7\sqrt{x}} \quad \text{and} \quad h(x) = \sqrt{x}.$$

If g is a function such that $f(x) = g(h(x))$, find a formula for $g(x)$.

Solution: Since $f(x) = \sqrt{1 + 7\sqrt{x}} = \sqrt{1 + 7h(x)}$, then $g(x) = \sqrt{1 + 7x}$.

4. [8 points] The number of people p (in thousands) who are sick with the flu virus t days after January 1, 2014 is given by

$$p = g(t) = \frac{3}{1 + e^{-0.3t}}$$

a. [4 points] Find a formula for $g^{-1}(p)$. Show all your steps to receive full credit.

Solution:

$$\begin{aligned} p &= \frac{3}{1+e^{-0.3t}} \\ p(1+e^{-0.3t}) &= 3 \\ 1+e^{-0.3t} &= \frac{3}{p} \\ e^{-0.3t} &= \frac{3}{p} - 1 \\ -0.3t &= \ln\left(\frac{3}{p}-1\right) \\ t &= \frac{1}{-0.3}\ln\left(\frac{3}{p}-1\right) \end{aligned}$$

b. [2 points] What is a practical interpretation of $g^{-1}(2)$? You do not need to compute its value. Include units.

Solution: It is the number of days after January 1, 2014 needed for two thousand individuals to be sick with the flu.

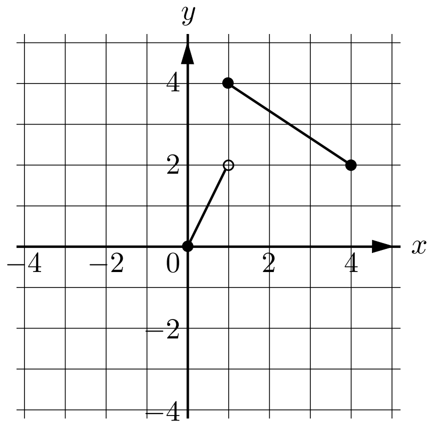
c. [2 points] The quantity of flu vaccine q (in liters) produced t days after January 1, 2014 is given by

$$q = f(t) = \frac{\sqrt{5}\, t^2}{(1+2t)^2}.$$

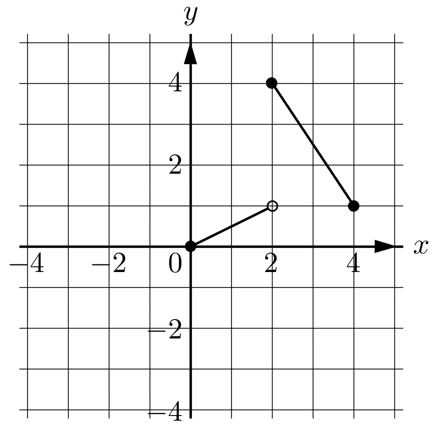
What eventually happens to the quantity of flu vaccine produced. Give your answer in **exact form**.

Solution: $\displaystyle \lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\sqrt{5}\, t^2}{(1+2t)^2} = \lim_{t\to\infty} \frac{\sqrt{5}\, t^2}{(2t)^2} = \lim_{t\to\infty} \frac{\sqrt{5}\, t^2}{4t^2} = \frac{\sqrt{5}}{4}.$

5. [8 points] The graph of the function f defined on the domain $[0, 4]$ is drawn below.



$y = f(x)$



$y = f^{-1}(x)$

a. [4 points] Using the axis above (labelled "$y = f^{-1}(x)$"), sketch the graph $y = f^{-1}(x)$.

b. [4 points] Write down a piecewise formula for the function f.

Solution:

$$f(x) = \begin{cases} 2x & 0 \le x < 1. \\ 4 - \frac{2}{3}(x-1) & 1 \le x \le 4. \end{cases}$$

6. [6 points] Let g be a function defined on the real line. Some values of g are shown below.

x	0	1	2	3
$g(x)$	0	5	6	7

a. [2 points] If g were an odd function, what should the value of $g(-1)$ be?

Solution: $g(-1) = -g(1) = -5$

b. [2 points] If g were a periodic function of period 5, what should the value of $g(-3)$ be?

Solution: $g(-3) = g(-3+5) = g(2) = 6$

c. [2 points] Let k be the function defined by $k(x) = g(2x+5)$. What is $k(-1)$?

Solution: $k(-1) = g(2(-1)+5) = g(3) = 7.$

7. [9 points] The total cost C (in thousands of dollars) for a farmer to grow p tons of potatoes is given by the function

$$C = h(p) = 2p^2 - 16p + 39.$$

a. [4 points] What is the maximum number of tons of potatoes the farmer can produce if he only has 35 thousand dollars to spend on growing potatoes? Your answer should be in **exact form**.

Solution:

$$\begin{aligned} 2p^2 - 16p + 39 &= 35 \\ 2p^2 - 16p + 4 &= 0 \\ p^2 - 8p + 2 &= 0 \end{aligned}$$

$$p = \frac{8 \pm \sqrt{64 - 4(2)}}{2} = \frac{8 \pm \sqrt{56}}{2} = 4 \pm \sqrt{14}$$

Number of tons of potatoes $= 4 + \sqrt{14}$.

b. [4 points] Complete the square to write the function h in vertex form. Show all your work step by step.

Solution:

$$\begin{aligned} 2p^2 - 16p + 39 &= 2(p^2 - 8p) + 39 \\ &= 2(p^2 - 8p + 16 - 16) + 39 \\ &= 2((p-4)^2 - 16) + 39 \\ &= 2(p-4)^2 - 32 + 39 = 2(p-4)^2 + 7 \end{aligned}$$

c. [1 point] How many tons of potatoes does the farmer need to produce in order to minimize the total cost?

Solution: Number of tons of potatoes= 4.

8. [9 points] A space ship has landed on Planet X. Scientists discovered that the surface temperature of Planet X oscillates sinusoidally between a maximum of $170°C$ to a minimum of $-40°C$. It takes 7 hours for the surface temperature to decrease from its maximum to its minimum. At the time the space ship landed, the surface temperature was $-40°C$. Let $P = g(t)$ be the surface temperature (in °C) of Planet X, t hours after the space ship landed.

a. [4 points] Find a formula for $g(t)$.

Solution: Maximum temperature=170°C and Minimum temperature=−40°C, hence

1.Midline: $P = k = \frac{Max+Min}{2} = \frac{170+(-40)}{2} = 65$.
2.Period=T=2(7)=14 hours.
3.Amplitude=$A = \frac{Max-Min}{2} = \frac{170-(-40)}{2} = 105$
4.Since the graph has a minimum at $t = 0$, then the function is given by a negative cosine cycle.

Hence using

$$g(t) = k - A\cos\left(\frac{2\pi}{T}t\right) = 65 - 105\cos\left(\frac{2\pi}{14}t\right) = 65 - 105\cos\left(\frac{\pi}{7}t\right)$$

The surface temperature K (in °C) of a moon of Planet X, t hours after the spaceship landed on Planet X, is given by the formula

$$K = Q(t) = 20 - 70\cos\left(\frac{2\pi}{3}t\right).$$

b. [5 points] Find the times in the interval $-1 \le t \le 3$ when the surface temperature of the moon is equal to 10° C. Your solutions should be in **exact form**.

Solution:

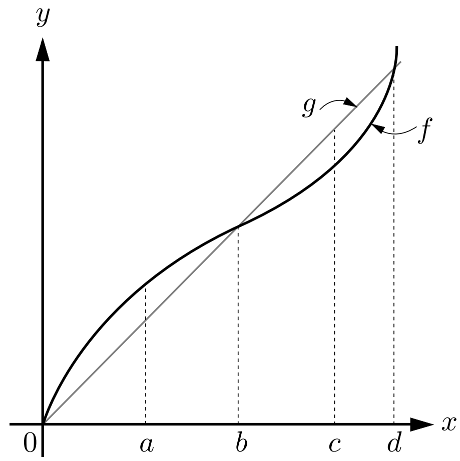
$$\begin{aligned} 20 - 70\cos\left(\frac{2\pi}{3}t\right) &= 10 \\ \cos\left(\frac{2\pi}{3}t\right) &= \frac{1}{7} \\ \frac{2\pi}{3}t &= \cos^{-1}\left(\frac{1}{7}\right) \\ t &= \frac{3}{2\pi}\cos^{-1}\left(\frac{1}{7}\right) \end{aligned}$$

Using the symmetries of the cosine function we get that in the interval $-1 \le t \le 3$, there are three solutions to the equation $t_1 < t_2 < t_3$.

$$t_2 = \frac{3}{2\pi}\cos^{-1}\left(\frac{1}{7}\right) \qquad t_1 = -t_2 \qquad t_3 = 3 - t_2.$$

9. [10 points]

a. [6 points] Ford and GM are two different automobile companies. It costs Ford $f(x)$ dollars and GM $g(x)$ dollars to produce x cars. The function g is given by the formula $g(x) = 4000x$, and the graphs of f and g are drawn below.



Indicate if each of the following statements are true or false by circling the correct answer. No justification is required.

1. The function f is invertible on the interval $[0, d]$.

   $\boxed{\text{True}}$ False

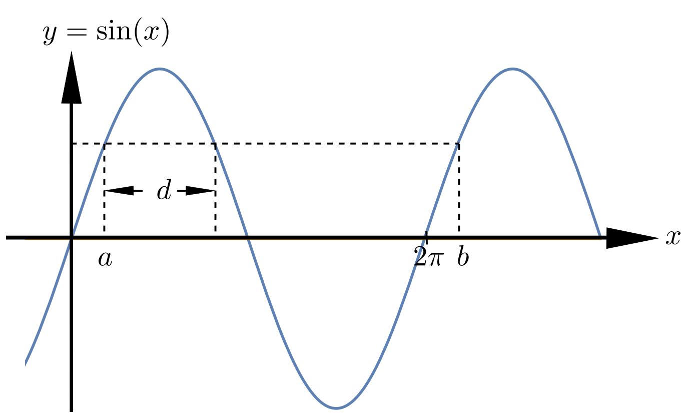
2. Suppose that GM and Ford both spend the same amount of money producing cars. Then GM will always produce more cars than Ford.

   True $\boxed{\text{False}}$

3. The average rate of change of the cost function for Ford between 0 and a is larger than the average rate of change of the cost function for Ford between 0 and c cars.

   $\boxed{\text{True}}$ False

b. [4 points] The graph of the function $f(x) = \sin(x)$ is drawn below, and the dotted lines are either horizontal or vertical. Find the following quantities only in terms of a. Simplify your answer as much as possible.



1. $\boxed{\text{The distance } d = \pi - 2a}$.

2. $\boxed{\sin^{-1}(\sin(b)) = \sin^{-1}(\sin(a)) = a}$.

10. [10 points] David and Harold each have some money that they deposit in a bank account at the same time. Let $D(t)$ and $H(t)$ be the amount of money (in dollars) in David's and Harold's bank accounts respectively, t years after they make the initial deposit. Harold's initial deposit of m dollars grows at a continuous annual rate of 10%, and $D(t) = 20,000(1.1)^t$.

a. [2 points] Find the growth factors of the functions D and H. Your answer must be in exact form.

Solution:
Growth factor for $D = 1.1$ Growth factor for $H = e^{0.1}$.

b. [2 points] Is the following statement true or false? If Harold's initial deposit is larger than David's, then there is some time after they made the initial deposit when David's and Harold's bank accounts have the same amount of money. Circle your answer.

True $\boxed{\text{False}}$

Solution:

c. [3 points] What should Harold's initial deposit be, in order for the amount of money in David's and Harold's bank accounts to be the same ten years after they made their initial deposits? Find your answer algebraically.Your answer must be exact or accurate up to the nearest cent.

Solution:

$$20,000(1.1)^{10} = me^{.1(10)}$$

$$\text{Harold's initial deposit } = m = \frac{20,000(1.1)^{10}}{e} \approx 19,083.69 \text{ dollars.}$$

d. [3 points] How many years does it take for David's initial deposit to triple? Find your answer algebraically. Your answer must be exact or accurate up to the first two decimals.

Solution:

$$\begin{aligned} 60,000 &= 20,000(1.1)^t \\ 3 &= (1.1)^t \\ \ln(3) &= t\ln(1.1) \\ t &= \frac{\ln(3)}{\ln(1.1)} \approx 11.52 \text{ years.} \end{aligned}$$

11. [15 points] The velociraptor population on the earth one year and four years after a huge meteor hits the earth is 2 million and 1.6 million respectively. Let P be the velociraptor population (**in millions**) on the earth t years after the meteor hits the earth.

a. [5 points] Suppose that the velociraptor population on the earth decreased exponentially after the meteor hits the earth. In this case, $P = g(t)$ for some function g. Find a formula for $g(t)$. Your answer should be in **exact form**.

Solution:

$$dg(t) = ab^t$$
$$2 = ab$$
$$1.6 = ab^4$$

$$\frac{ab^4}{ab} = \frac{1.6}{2} = 0.8$$
$$b^3 = 0.8 \qquad b = (0.8)^{\frac{1}{3}} \qquad a = \frac{2}{b} = \frac{2}{(0.8)^{\frac{1}{3}}}$$

$$g(t) = \frac{2}{(0.8)^{\frac{1}{3}}}\left((0.8)^{\frac{1}{3}}\right)^t$$

b. [4 points] Suppose that the velociraptor population on the earth is a power function of t, the number of years after the meteor hits the earth. In this case, $P = h(t)$ for some function h. Find a formula for $h(t)$. Your answer should be in **exact form**.

Solution:

$$h(t) = kt^p$$
$$2 = k(1)^p = k$$
$$1.6 = k(4)^p$$

$$1.6 = 24^p$$
$$4^p = 0.8$$
$$\ln(4^p) = \ln(0.8)$$
$$p\ln(4) = \ln(0.8)$$
$$p = \frac{\ln(0.8)}{\ln(4)} \qquad h(t) = 2t^{\frac{\ln(0.8)}{\ln(4)}}$$

The problem continues on the next page

The velociraptor population on the earth one year and four years after a huge meteor hits the earth is 2 million and 1.6 million respectively. Let P be the velociraptor population (**in millions**) on the earth t years after the meteor hits the earth.

c. [1 point] Under which assumption does P decrease faster to 0, if we assume that $P = g(t)$ or if we assume that $P = h(t)$? Circle your answer.

Solution:

$\boxed{P = g(t)}$ $\quad\quad$ $P = h(t)$ $\quad\quad$ Cannot be determined.

d. [3 points] Suppose that the velociraptor population on the earth decreased linearly after the meteor hits the earth. In this case, $P = f(t)$ for some function f. Find a formula for $f(t)$.

Solution: $m = \dfrac{2 - 1.6}{1 - 4} = -\dfrac{0.4}{3}$, then $f(t) = 2 - \dfrac{0.4}{3}(t - 1)$.

e. [2 points] Give a practical interpretation of the horizontal intercept of the graph $P = f(t)$.

Solution: The number of years after the meteor hit earth needed to eradicate the population of velociraptors.

12. [6 points] Let $N(x)$ be the cost (in dollars) to produce x pieces of chocolate. The chocolates are then put into boxes containing ten pieces of chocolate each. The packaging costs for each box of chocolates is \$0.15. Write down a mathematical expression describing the following.

a. [2 points] The average cost (in dollars per piece of chocolate) of producing c chocolates.

Solution: $\dfrac{N(c)}{c}$

b. [2 points] The cost in dollars of producing the fifteenth piece of chocolate.

Solution: $N(15) - N(14)$

c. [2 points] The total cost in dollars (including packaging costs) of producing b boxes of chocolate.

Solution: $0.15b + N(10b)$.