Math 105 — First Midterm

October 12, 2015

Name: EXAM SOLUTIONS

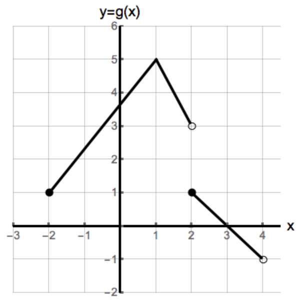
Instructor: __________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	22	
2	11	
3	13	
4	10	
5	12	
6	11	
7	9	
8	12	
Total	100	

1. [22 points] Consider the functions $f(x)$, $g(x)$ and $h(x)$ given below:

x	-1	0	1	2
$f(x)$	-5	6	2	1



$h(x) = 2x - 7$

a. [4 points] Find the domain and range of $g(x)$. Use inequalities or interval notation to express your answers.

Solution: Domain: $[-2, 4)$, Range: $(-1, 5]$

b. [11 points] Find the values of the following expressions. If any of the values is not defined, write "Undefined".

Solution:

i) $g(2) = 1$
ii) $(f(0))^{-1} = 1/6$
iii) $h^{-1}(2) = 9/2$
iv) $g(f(2)) = 5$
v) $g(g(-2)) = 5$
vi) $h(f(-1) + 1) = -15$

c. [4 points] Find all solutions to the equation $h(g(x)) = -5$. Recall that $h(x) = 2x - 7$ and the graph of g has been copied below in part d) for your convenience.

Solution:

$$h(g(x)) = -5$$

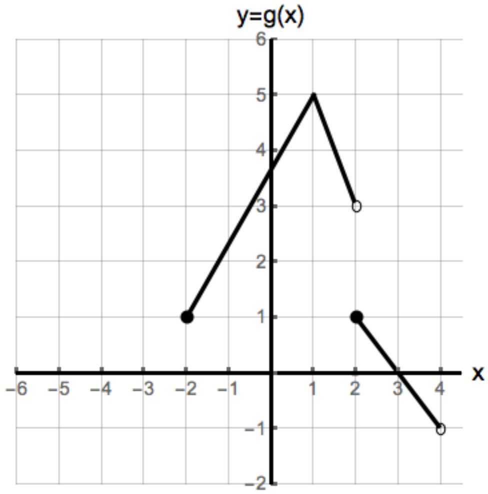
$$2g(x) - 7 = -5$$

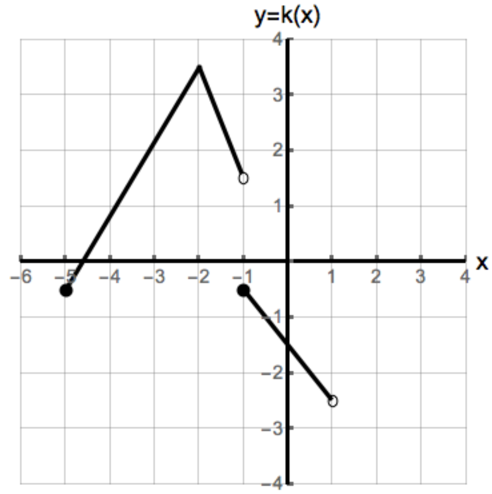
$$2g(x) = 2$$

$$g(x) = 1$$

Answer: $x = \pm 2$

d. [3 points] Find a formula for $k(x)$ in terms of the function g.





Solution: To obtain the graph of $k(x)$ we need to perform a vertical shift 1.5 units down and a shift left 3 units. Hence,

$$k(x) = g(x + 3) - 1.5$$

2. [11 points] The Wasem Fruit Farm produces and sells apples to its visitors during the Fall.

i) Let $f(t)$ be the number of apples sold at the Wasem Fruit Farm t days after September 10.

ii) The revenue (in dollars) obtained by the Wasem Fruit Farm from selling a apples is given by the function $g(a)$.

A local diner produces hot apple cider.

i) Let $h(p)$ be the number of gallons of hot apple cider produced by the diner with p apples.

ii) The revenue (in dollars) obtained by the diner from selling c gallons of hot apple cider is given by the function $j(c)$

Assume that all the functions defined above have an inverse.

a. [5 points] Write a practical interpretation of the following mathematical expressions:

Solution:

i) On September 20, the revenue of Wasem Fruit Farm was $199.

ii) $g^{-1}(20)$ is the number of apples giving a revenue of $20.

b. [3 points] Let a_0 be the average amount of apples sold in a day by Wasem Fruit Farm. The function $Q(y)$ gives the number of gallons of hot apple cider the diner can produce with y more apples than the average amount sold in a day by the farm. Find a formula for $Q(y)$ in terms of the functions defined above.

Solution:

$$Q(y) = h(y + a_0)$$

c. [3 points] Write down an equation that represents the following statement:

The revenue obtained by Wasem Fruit Farm on September 25 is equal to the revenue obtained by the diner for the sale of 21 gallons of hot apple cider.

Solution:

$$g(f(15)) = j(21)$$

3. [13 points] A berry crop in Michigan has been invaded by fruit flies since 2010. In that year, it was estimated that there were 33 million fruit flies on the crop. Doctor Banner has been investigating the infestation and he discovered that the population of fruit flies increases exponentially. His records show that the population of fruit flies increased by 30% of its original size in the time period between 2010 and 2015.
Let $s(t)$ be the function which gives the number of fruit flies (in millions) in the crop t years after 2010. Your answers in parts a), b) and c) must be **exact** or accurate up to the first 4 decimals.

a. [4 points] What is the annual percent growth rate of $s(t)$?

Solution:

$$b^5 = 1.3$$
$$b = \sqrt[5]{1.3}$$
$$r = b - 1 = \sqrt[5]{1.3} - 1$$

Annual percent growth rate of $s(t)$=$\sqrt[5]{1.3} - 1 \approx 0.0538$

b. [2 points] Find a formula for the function $s(t)$.

Solution:

$$s(t) = 33(1.3)^{t/5}$$

c. [2 points] According to your formula for $s(t)$, how many fruit flies will there be in the crop in 2017?

Solution:

$$s(7) = 33(1.3)^{7/5} \text{ millions}$$

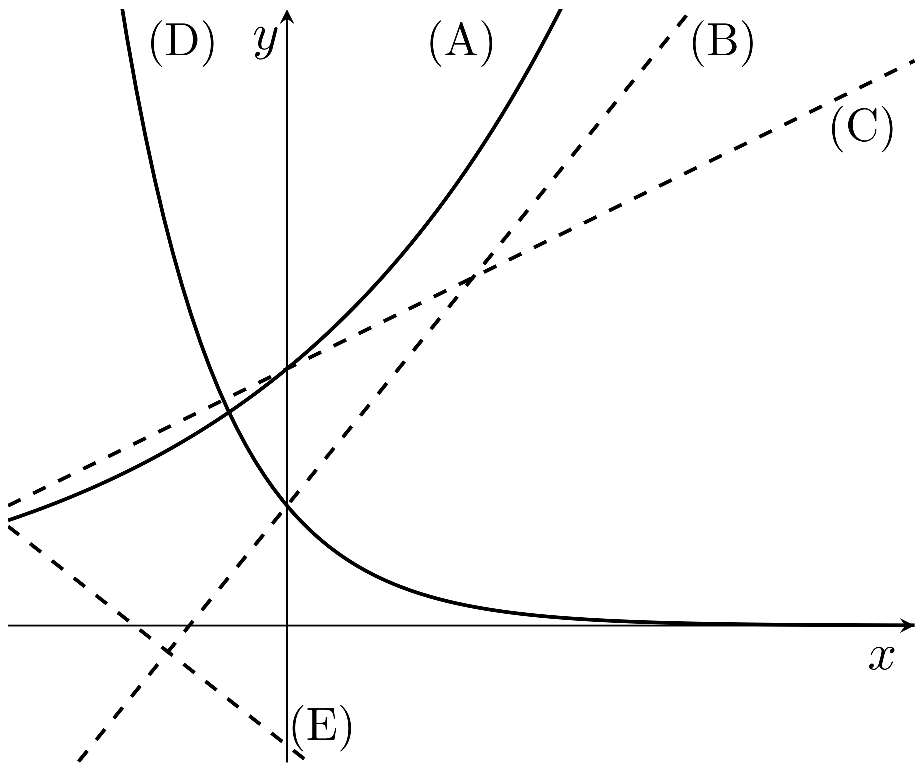
d. [5 points] Find the average rate of change of $s(t)$ between 2010 and 2017 and interpret your result in the context of this problem.

Solution: Average rate of change=

$$\text{Average rate of change} = \frac{33(1.3)^{7/5} - 33}{7} = 2.0924$$

On average, between 2010 and 2017, the number of fruit flies in the crop will increase by 2.0924 millions every year.

4. [10 points] The following picture shows the graphs of five different functions. Below the graphs, you are also given the formulas for these functions, where p, q, r, s are some positive numbers.



Match each formula to the respective graph A, B, C,D or E.

a. [2 points] $y = p + sx$ ____________________

Solution: (C)

b. [2 points] $y = q + rx$ ____________________

Solution: (B)

c. [2 points] $y = -q - \frac{1}{r}x$ ____________________

Solution: (E)

d. [2 points] $y = pr^x$ ____________________

Solution: (A)

e. [2 points] $y = qs^x$ ____________________

Solution: (D)

5. [12 points] Three researchers Dr. Banner, Dr. Storm and Dr. Kyle are studying a population of an alien species, commonly known as CATS (Category A Threats). The study of CATS started in 2010 and the scientists observed a population of 125 CATS in 2012 and a population of 600 CATS in 2014. In all the functions below, the variable t represents the number of *years after 2010*. Show all your work.

a. [3 points] Doctor Banner believes the CATS population is described by the linear function $R(t)$. Find a formula for $R(t)$.

Solution: Slope:

$$m = \frac{600 - 125}{4 - 2} = 237.5$$

Point-Slope formula:

$$B(t) = 125 + 237.5(t - 2)$$

$$B(t) = 237.5t - 350$$

b. [4 points] Doctor Storm thinks that a linear model is not adequate to describe the population of CATS. She believes that the number of CATS can be described by a quadratic function $S(t)$ whose minimum occurred in 2012. Find a formula for $S(t)$.

Solution: Vertex at $(2, 125)$, so

$$S(t) = a(t - 2)^2 + 125.$$

Plug in $(4, 600)$ to solve for a.

$$600 = a(4 - 2)^2 + 125$$

$$475 = 4a$$

$$a = 118.75$$

Hence $S(t) = 118.75(t - 2)^2 + 125$

c. [5 points] On the other hand, Doctor Kyle strongly believes that the CATS' population size must grow exponentially. He describes the population of CATS using the exponential function $K(t)$. Find a formula for $K(t)$. Your answer must be in **exact form**.

Solution: Find b. $600 = ab^4$ and $125 = ab^2$ yields $b^2 = 4.8$. Hence $b = \sqrt{4.8}$. Plug in a point and solve for a.

$$125 = a(\sqrt{4.8})^2$$

$$125 = 4.8a, \qquad a = \frac{125}{4.8}$$

Hence,

$$K(t) = \frac{125}{4.8}(4.8)^{t/2}$$

6. [11 points] Ammonia is leaked from a boat into a lake. The initial amount of ammonia spilled is 500 gallons. Include units in all your answers.

a. [3 points] After the spill started, the amount of ammonia in the lake increased at a rate of 5 percent every minute. It took the crew in the boat 30 minutes to access and stop the leak from the boat. How much ammonia was spilled in the lake by the time the leak was repaired? Your answer must be in **exact form**.

Solution:
$500(1.05)^{30}$ gallons

b. [2 points] Right after the leak was stopped, the chemical cleaning crew started to remove the ammonia from the lake. They are able to remove 100 gallons of ammonia from the lake every 5 minutes. How long does it take for the cleaning crew to remove all the ammonia from the lake? Your answer must be in **exact form**.

Solution: Every minute $\frac{100}{5} = 20$ gallons of ammonia are removed. Hence, it takes $\dfrac{500(1.05)^{30}}{20} = 25(1.05)^{30}$ minutes to remove all the ammonia.

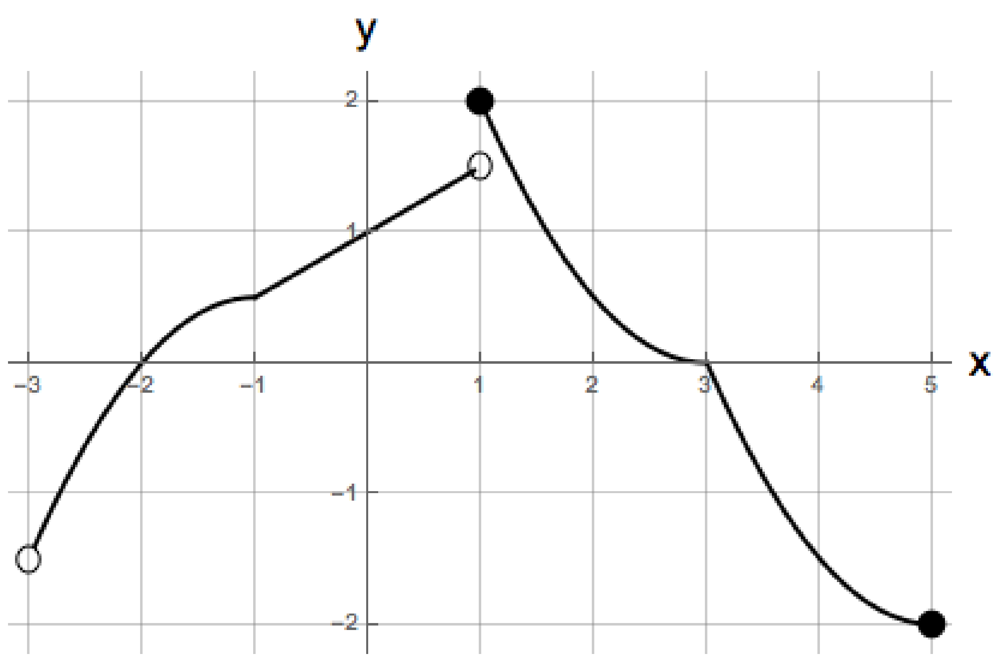
c. [6 points] Let $Q(t)$ be the amount of ammonia in the lake (in gallons) t minutes after the spill started. The function $Q(t)$ is only defined from the time the spill started until all the ammonia is removed from the lake by the cleaning crew. Find a piecewise defined formula for $Q(t)$. Your answer must be in **exact form**. Show all your work.

Solution:

$$Q(t) = \begin{cases} 500(1.05)^t & 0 \le t \le 30 \\ -20(t-30) + 500(1.05)^{30} & 30 < t \le 25(1.05)^{30} + 30 \end{cases}$$

7. [9 points] On the axes provided below, sketch the graph of **one possible function** $y = g(x)$, satisfying **all** of the following requirements. Your graph should clearly show the properties listed below to receive full credit.

- ☐ The domain of g is $(-3, 5]$.
- ☐ The range of g is $[-2, 2]$.
- ☐ g has vertical intercept $(0, 1)$.
- ☐ g has exactly two zeros, at $x = -2$ and at $x = 3$.
- ☐ g has a constant rate of change for $-1 < x < 1$.
- ☐ g is increasing for $x < 0$.
- ☐ g is concave up for $x > 3$.
- ☐ g attains its minimum value at $x = 5$.



8. [12 points] Let

$$V(x) = -\frac{1}{2}x^2 + \frac{9}{2}x + \frac{47}{8} \qquad \text{for} \qquad 0 \leq x \leq 10$$

be the number of viewers of a 10-minute interview (in millions), x minutes after the interview started.

a. [5 points] Write the quadratic function $V(x)$ in vertex form by completing the square. Show all your work carefully, step by step to receive full credit.

Solution:

$$V(x) = -\frac{1}{2}(x^2 - 9x) + \frac{47}{8}$$
$$V(x) = -\frac{1}{2}\left(x^2 - 9x + \left(\frac{9}{2}\right)^2 - \left(\frac{9}{2}\right)^2\right) + \frac{47}{8}$$
$$V(x) = -\frac{1}{2}\left(\left(x - \frac{9}{2}\right)^2 - \frac{81}{4}\right) + \frac{47}{8}$$
$$V(x) = -\frac{1}{2}\left(x - \frac{9}{2}\right)^2 + \frac{81}{8} + \frac{47}{8}$$
$$V(x) = -\frac{1}{2}\left(x - \frac{9}{2}\right)^2 + 16$$

b. [3 points] In how many minutes after the beginning of the interview did the number of viewers reach its minimum and maximum, respectively?

Solution: The minimum occurs at one of the endpoints. One checks that it occurs at $x = 10$. The maximum occurs at the x-coordinate of the vertex i.e., at $x = 4.5$.

c. [4 points] For how long will the number of viewers of the interview be more than 10 million? Recall that

$$V(x) = -\frac{1}{2}x^2 + \frac{9}{2}x + \frac{47}{8} \qquad \text{for} \qquad 0 \leq x \leq 10.$$

Solve this problem algebraically. Your answer must be in **exact form**. Show all your work.

Solution: $-0.5x^2 + 4.5x + \frac{47}{8} = 10$ or $-0.5x^2 + 4.5x - 4.125 = 0$
Using the quadratic formula we get two solutions in the function's domain:

$$x = 4.5 \pm \sqrt{(4.5)^2 - 4(-0.5)(-4.125)} = 4.5 \pm 2\sqrt{3}.$$

Hence the number of viewers is larger than 10 millions for $4\sqrt{3}$ minutes.