Math 105 — Second Midterm

November 16, 2015

Name: EXAM SOLUTIONS

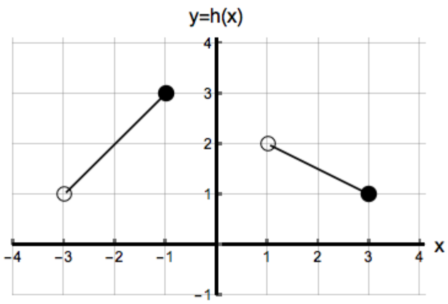
Instructor: ______________________ Section: ____________

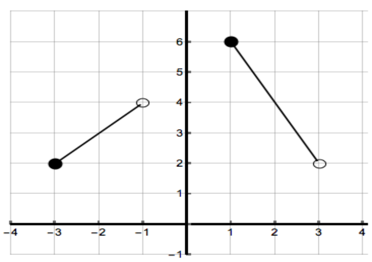
1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

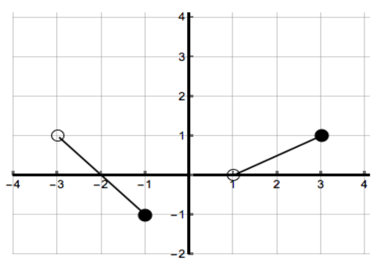
Problem	Points	Score
1	15	
2	14	
3	8	
4	9	
5	10	
6	10	
7	12	
8	8	
9	6	
10	8	
Total	100	

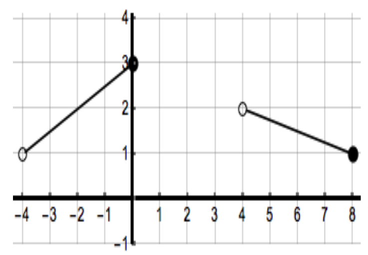
1. [15 points]

a. [9 points] The graph of the function $y = h(x)$ is shown below. The other graphs below it can be obtained by applying transformations to the graph of $y = h(x)$. Write the letter that corresponds to the correct function in the line below each graph. If the correct answer is not listed below, write the correct formula on the line provided below each graph.









Solution: G, J, D

A) $y = h(2x+1)$ B) $y = -2h(x)$ C) $y = -h(x) - 2$ D) $y = h\left(\frac{1}{2}x - 1\right)$

E) $y = h(-2x)$ F) $y = h(2x-1)$ G) $y = 2h(-x)$ H) $y = -2h(-x)$

I) $y = h\left(\frac{3}{2}x + 1\right)$ J) $y = 2 - h(x)$ K)$y = h\left(\frac{1}{2}x + 1\right)$ L) $y = h(x) - 2$

b. [6 points]

Solution: Compute the value of the following limits:

i) $\lim_{x \to -\infty} 3 + e^{-x^2} = 3$

ii) $\lim_{x \to 0^-} 4\ln(-x) = -\infty$

iii) $\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$

2. [14 points]

a. [10 points] Let $f(c)$ be Lucy's revenue (in dollars) when she sells c eggs at the farmers market. Let c_0 be the number of eggs she sold on Saturday. Write a mathematical expression that completes each of the following statements. All your answers should be in terms of the function f.

Solution:

i) Lucy's revenue, in dollars, when she sells 25% more eggs than she sold on Saturday is

Answer: $f(1.25c_0)$

ii) Mark is another farmer selling eggs at the market. Mark's revenue on Saturday was 10 dollars less than Lucy's revenue that day. On Saturday Mark's revenue, in dollars, was

Answer: $f(c_0) - 10$

iii) On Wednesday, Lucy sold 10 more eggs than on Saturday. Lucy's revenue on Wednesday, in dollars, was

Answer: $f(c_0 + 10)$

iv) Let $g(d)$ be Lucy's revenue in **hundreds** of dollars when she sells d **dozen** eggs, then

$$g(d) = 0.01f(12d)$$

b. [4 points] Find the equations of the horizontal and vertical asymptotes of each function below. If the given function does not have one of the asymptotes, write "NONE".

Solution:

i) $y = 3(0.21)^{-2x}$

Horizontal Asymptote: $y = 0$ Vertical Asymptote: NONE

ii) $y = 1 + \ln(0.2x + 1)$

Horizontal Asymptote: NONE Vertical Asymptote: $x = -5$

3. [8 points] Let $S(t)$ be the amount of shrimp (in thousands) living in a lake t years after January 1, 2000, where

$$S(t) = 3.27(1.3)^t.$$

a. [3 points] In how many years, after January 1, 2000, will the number of shrimps in the lake have increased by 75%? Your answer must be exact or accurate up to the first two decimals.

Solution:

$$\begin{aligned} 3.27(1.3)^t &= 1.75(3.27) \\ (1.3)^t &= 1.75 \\ t\ln(1.3) &= \ln(1.75) \\ t &= \frac{\ln(1.75)}{\ln(1.3)}. \end{aligned}$$

b. [2 points] What is the continuous growth rate per year of the population of shrimps? Your answer must be exact or accurate up to the first two decimals.

Solution: $k = \ln(1.3)$

c. [3 points] Let $f(p)$ be the amount of shrimps, in thousands, p **months** after January 1, 2000. What is the growth factor of the function $f(p)$? Your answer must be in **exact** form.

Solution: $f(p) = S\left(\frac{p}{12}\right) = 3.27(1.3)^{\frac{p}{12}}$. Growth factor of $f(p)$ is $b = (1.3)^{\frac{1}{12}}$.

4. [9 points] The $\mathcal{M}$-scale M of an object in outer space with diameter D, in thousands of miles, is given by

$$M = f(D) = 2 + 11.5\log\left(\frac{D}{d_0}\right)$$

where d_0 is a positive constant.

a. [4 points] If the $\mathcal{M}$-scale of a planet is 10, what is its diameter? Solve algebraically showing all your steps. Your answer may depend on the constant d_0.

Solution:

$$\begin{aligned}
2 + 11.5\log\left(\frac{D}{d_0}\right) &= 10\\
11.5\log\left(\frac{D}{d_0}\right) &= 8\\
\log\left(\frac{D}{d_0}\right) &= \frac{8}{11.5}\\
\frac{D}{d_0} &= 10^{\frac{8}{11.5}}\\
D &= 10^{\frac{8}{11.5}} d_0
\end{aligned}$$

b. [5 points] Let D_B and D_M be the diameters of two planets, planet Blue and planet Maize, respectively. If the diameter of planet Blue is double the diameter of planet Maize, then what is the difference between $\mathcal{M}$-scale values of planet Blue and planet Maize? Show all your computations step by step. *Simplify your answer as much as possible.*

Solution:

$$\begin{aligned}
M_{Blue} - M_{Maize} &= 2 + 11.5\log\left(\frac{D_B}{d_0}\right) - \left(2 + 11.5\log\left(\frac{D_B}{d_0}\right)\right)\\
&= 11.5\log\left(\frac{D_B}{d_0}\right) - 11.5\log\left(\frac{D_M}{d_0}\right)\\
&= 11.5\left(\log\left(\frac{D_B}{d_0}\right) - \log\left(\frac{D_M}{d_0}\right)\right)\\
&= 11.5\log\left(\left(\frac{D_B}{d_0}\right)\left(\frac{d_0}{D_M}\right)\right) = 11.5\log\left(\frac{D_B}{D_M}\right)\\
&= 11.5\log\left(\frac{2D_M}{D_M}\right) = 11.5\log(2)
\end{aligned}$$

5. [10 points] Indicate if each of the following statements is true or false by circling the correct answer. No justification is required.

a. [2 points] The graph of $y = \ln\left(\frac{1}{x}\right)$ can be obtained by applying transformations to the graph of $y = \ln(x)$.

$\boxed{\text{True}}$ False

Solution:

b. [2 points] The function $y = f(x)$ has domain $[-2, 1]$, then the graph of $y = 2f(3x) + 1$ has a domain $[-6, 3]$.

True $\boxed{\text{False}}$

Solution:

c. [2 points] If a population has a half life of two years, then after six years the population decreases by 87.5%.

$\boxed{\text{True}}$ False

Solution:

d. [2 points] If the hypotenuse of a right triangle has length 5 meters and one of its sides has length 4 meters, then the tangent of one of the angles of the triangle has to be $\frac{4}{5}$.

True $\boxed{\text{False}}$

Solution:

e. [2 points] If a periodic function f with domain $(-\infty, \infty)$ has period 5, then $f(2) = f(22)$.

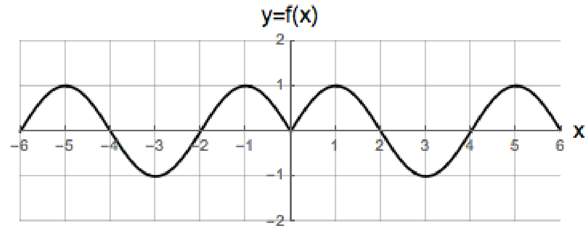
$\boxed{\text{True}}$ False

Solution:

6. [10 points] Consider the functions $f(x)$, $h(x)$ and $q(x)$

x	-3	-2	-1	0	1	2	3
$q(x)$	15	10	5	10	15	10	5

$$h(x) = \frac{-x}{1+x^2}.$$



a. [2 points] Suppose that one of the functions above is a periodic function whose period is an integer less than six. Find the periodic function and determine its period.

Solution: The function $q(x)$ is periodic with period 4.

b. [2 points] Which of the functions above are odd? Circle all that apply.

Solution:

$q(x)$ $\boxed{h(x)}$ $f(x)$ NONE

c. [2 points] Which of the functions above are even? Circle all that apply.

$q(x)$ $h(x)$ $\boxed{f(x)}$ NONE

Solution:

d. [4 points] Consider an even function $y = p(x)$ that has range $[-2, 2]$ and the function $g(x) = 5p(-2x) + 1$.

Solution:

i) What is the range of $g(x)$? Write your answer using inequalities or interval notation.

Range of $g(x)$: $[-9, 11]$

ii) Is $g(x)$ even, odd or neither? Circle your answer.

$\boxed{\text{EVEN}}$ ODD NEITHER

7. [12 points] Solve for x the following equations algebraically. Show all your work step by step and write your answers in **exact form** to receive full credit.

a. [4 points] $4(10^{2\log(x)+1}) = 3$

Solution:

$$\begin{aligned} 4(10^{2\log(x)+1}) &= 3 \\ 10^{2\log(x)+1} &= 0.75 \\ 2\log(x)+1 &= \log(0.75) \\ \log(x) &= \frac{\log(0.75)-1}{2} \\ x &= 10^{\frac{\log(0.75)-1}{2}} \end{aligned}$$

b. [4 points] In this problem k is a constant, hence your answer may depend on k.

$$e^{kx} = 2e^{x+2}$$

Solution:

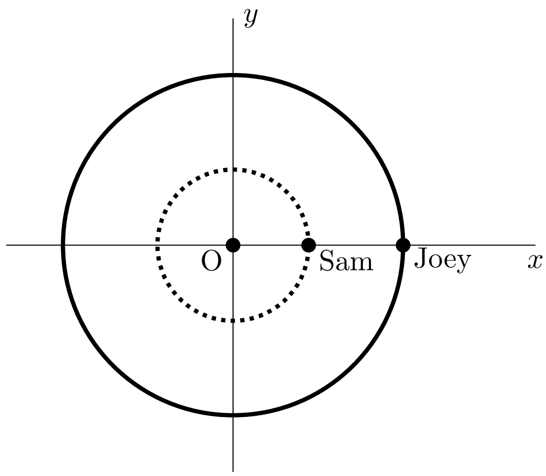
$$\begin{aligned} e^{kx} &= 2e^{x+2} \\ kx &= \ln(2e^{x+2}) \\ kx &= \ln(2) + \ln(e^{x+2}) \\ kx &= \ln(2) + x + 2 \\ kx - x &= \ln(2) + 2 \\ (k-1)x &= \ln(2) + 2 \\ x &= \frac{\ln(2)+2}{k-1}. \end{aligned}$$

c. [4 points] $\log(100x) = 2 + 2\log(x^2)$

Solution:

$$\begin{aligned} \log(100x) &= 2 + 2\log(x^2) \\ \log(100) + \log(x) &= 2 + 4\log(x) \\ 2 + \log(x) &= 2 + 4\log(x) \\ \log(x) &= 0 \\ x &= 1 \end{aligned}$$

8. [8 points] Three friends decide to go to a circular running track. Joey and Sam will run on the track and Holly will be timing them. Joey runs on the largest circle which has radius 250 meters. Sam runs on the inner most circle which has radius 100 meters.



The run will last 5 minutes, timed by Holly. At Holly's signal, Joey and Sam will both start from the most eastern point of their circle. Assume that both Joey and Sam run at a constant velocity at all times during the 5 minutes and that they both run in the counterclockwise direction.

a. [2 points] Holly notices that it takes Joey 4 minutes to run around the track once. What is the angle θ, in **radians**, that Joey forms with the starting position at the end of the 5 minute run?

Solution: Every minute Joey runs a quarter of the circumference of his track. Hence at the end of 5 minutes, he would have covered an angle of $\frac{5}{2}\pi$ radians.

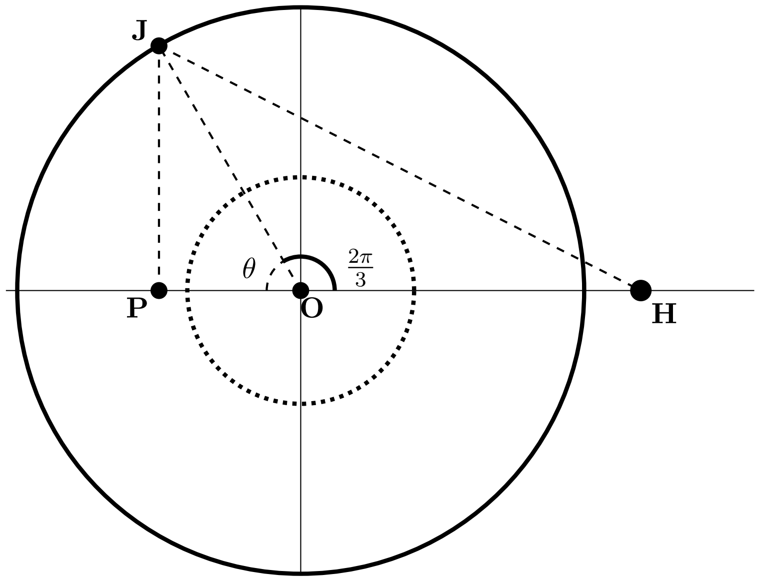
b. [4 points] On the other hand, Holly records that in one minute Sam covers $\dfrac{7}{8}$ of her track. Imagine that the origin of a coordinate system is placed at the center of the tracks, and that the x-axis is the west-east direction, whilst the y-axis is the south-north direction, as shown in the diagram. Find the coordinates of Sam's final position at the end of the 5 minutes.

Solution: After 5 minutes, Sam covers an angle of $5\left(\frac{7}{8}\right)(2\pi) = \frac{35}{4}\pi$ radians. Hence the coordinates of her final position on the circle of radius 100 are $\left(100\cos\left(\frac{35}{4}\pi\right), 100\sin\left(\frac{35}{4}\pi\right)\right) = (-50\sqrt{2}, 50\sqrt{2})$

c. [2 points] Compute the distance ran by Sam during the 5 minutes of the run.

Solution: At the end of the race, Sam ran a distance of $s = 100\left(\frac{35}{4}\pi\right) = 875\pi$ meters.

9. [6 points] Joey and Sam are running on circular tracks, timed by Holly. Joey runs on the largest circle which has radius 250 meters. Sam runs on the inner most circle which has a radius 100 meters. Suppose that Holly is standing 300 meters east of the center of the track and the point H in the diagram below indicates her position. Joey's position is indicated by the point J. The line JP is perpendicular to the horizontal line passing through the center O of the track.



a. [1 point] Suppose that the angle HOJ measures $\frac{2\pi}{3}$ radians, as shown in the diagram. What is the measure, in radians, of θ (the angle JOP in the diagram)?

Solution: $\theta = \frac{\pi}{3}$ radians.

b. [2 points] What is the length of the line segment JP?

Solution: JP$= 250 \sin\left(\frac{\pi}{3}\right) = 125\sqrt{3}$ meters.

c. [3 points] What is the length of the line segment HJ?

Solution:
HJ$=\sqrt{(250\sin\left(\frac{\pi}{3}\right))^2 + (300 + 250\cos\left(\frac{\pi}{3}\right))^2} = \sqrt{(125\sqrt{3})^2 + (300+125)^2} \approx 476.96$ m.

10. [8 points]

a. [4 points] The periodic function $y = f(t)$ gives the height, in meters above sea level, of the tide t hours after noon in Florence. The maximum height of the tide is called a high tide, whilst the minimum height of a tide is called a low tide. In Florence a high tide of 0.2 meters above sea level occurred at 2 pm, while a low tide of -0.8 meters (0.8 meters below sea level) will occur at 8 pm. Find the amplitude and the midline of $f(t)$.

Solution:
Amplitude of $f(t) = \frac{0.2-(-0.8)}{2} = 0.5$ Midline of $f(t)$: $y = \frac{0.2+(-0.8)}{2} = -0.3$

b. [4 points] The function $g(x)$ gives the height, in meters above sea level, of the tide x hours after noon in Edinburgh. Edinburgh is on the GMT time zone, so it is one hour behind Florence. The graph of $y = g(x)$ has:

i) Amplitude equal to 2.5 meters.
ii) Midline $y = 1.25$.

If the high tides and low tides times match across the globe (for example if a high tide occurs in Florence at 2 pm, then a high tide occurs in Edinburgh at 1 pm) and the graph of $g(x)$ can be obtained by applying transformations to the graph of f, write a formula for $g(x)$ in terms of the function f.

Solution: $g(x) = 5f(x+1) + 2.75$.