Math 105 — Final Exam

December 17, 2015

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	14	
3	6	
4	13	
5	6	
6	10	
7	10	
8	8	
9	12	
10	10	
Total	100	

1. [11 points] An animal shelter takes care of abandoned cats and dogs. Consider the following functions and constants:

- The function $F(c)$ gives the amount of pounds of cat food consumed by c cats in one day at the animal shelter.
- The function $S(p)$ gives the cost (in dollars) of p pounds of cat food.
- There were k cats in the animal shelter on December 17.
- On December 17, the animal shelter spent d dollars on dog food.

a. [4 points] Find a practical interpretation for each of the following mathematical expressions.

Solution:

i) $S^{-1}(13)$: The number of pounds of cat food that cost 13 dollars.

ii) $S(F(15))$: The cost in dollars of feeding 15 cats in one day at the animal shelter.

b. [7 points] Write a mathematical expression for each of the following quantities.

i) The average amount of cat food needed per cat in one day if there are c cats in the animal shelter.

Solution: $\frac{F(c)}{c}$

ii) The cost (in **hundreds** of dollars) of z **ounces** of cat food (recall that 1 pound equals 16 ounces).

Solution: $\frac{1}{100}S\left(\frac{z}{16}\right)$

iii) The amount of dollars the animal shelter spent on dog **and** cat food on December 17.

Solution: $d + S(F(k))$

2. [14 points]

a. [3 points] The population of aliens on planet Maize increases at a constant rate of 10 aliens every two years. We know that in 2005 there were 120 aliens on planet Maize. Find a formula for $M(t)$, the function which gives the number of aliens on planet Maize t *years after 2000*.

Solution: $M(t) = 5(t-5) + 120 = 5t + 95$

b. [3 points] Suppose that the population of aliens on planet Yellow in any given year is a thousand more the population of aliens on planet Maize ten years earlier. Find a formula for $Y(t)$, the population of planet Yellow t *years after 2000*, in terms of the function M.

Solution: $Y(t) = M(t-10) + 1000$

c. [3 points] The population of aliens on the planet Blue decreases at a continuous percent rate of 10 % per year. We know that in 2002 there were 100 aliens on planet Blue. Find a formula for $B(t)$, the function which gives the number of aliens on planet Blue t *years after 2000*.

Solution: $B(t) = \frac{100}{e^{-0.2}} e^{-0.1t} \approx 122.14 e^{-0.1t}$

d. [5 points] The alien population on planet Navy t *years after 2000* is given by the function $N(t)$, where

$$N(t) = \frac{100}{1+t^2}.$$

Find the average rate of change of $N(t)$ over the interval $[1,3]$ and give a practical interpretation of your result.

Solution:

$$\frac{N(3) - N(1)}{2} = \frac{10 - 50}{2} = -20$$

Between 2003 and 2001 the alien population on Planet Navy decreased on average by 20 aliens per year.

3. [6 points] The values of the functions $f(x)$, $g(x)$ and $h(x)$ are given below.

x	0	4	8	12
$f(x)$	100	20	4	0.8
$g(x)$	3.6	4.7	5.8	6.9
$h(x)$	-4	-3.6	-3	-0.9

a. [2 points] Which of the following functions could be linear? Circle all that apply.

Solution:
$f(x)$ $\boxed{g(x)}$ $h(x)$ None of these

b. [2 points] Which of the following functions could be exponential? Circle all that apply.

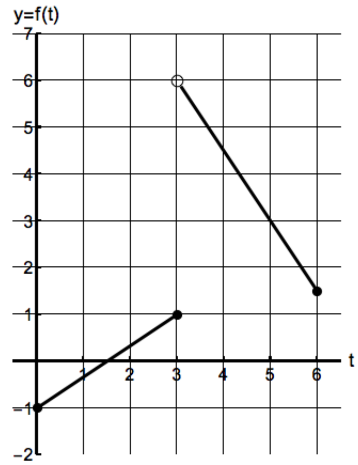
Solution:
$\boxed{f(x)}$ $g(x)$ $h(x)$ None of these

c. [2 points] Which of the following functions could be concave up? Circle all that apply.

Solution:
$\boxed{f(x)}$ $g(x)$ $\boxed{h(x)}$ None of these

4. [13 points]

a. [7 points] The graph of the function $f(t)$ is shown below



i) Find a formula for $f(t)$.

Solution:

$$f(t) = \begin{cases} \frac{2}{3}t - 1 & 0 \le t \le 3 \\ -\frac{3}{2}(t-5) + 3 = -1.5t + 10.5 & 3 < t \le 6 \end{cases}$$

ii) Does the function $f(t)$ have an inverse function for $0 \le t \le 6$? Circle your answer.

Solution:

$\boxed{YES}$ NO It is not possible to be determined.

b. [6 points] Find the value of the following limits.

Solution:

i) $\lim\limits_{x\to\infty} \dfrac{100\ln(100x)}{x^{0.2}} = 0$

ii) $\lim\limits_{x\to\infty} \dfrac{x^2(5-x^3)}{3+2x^5+6x^2} = -0.5$

iii) $\lim\limits_{\mathbf{x}\to -\infty} \dfrac{5+10^x}{3^x+7} = \dfrac{5}{7}$

5. [6 points] Let

$$F(x) = \frac{(x^2 - 4x + 4)x}{x(x - 100)(10x - 2)}$$

i) Does the graph of $y = F(x)$ have any vertical asymptotes? If so, write their equations, otherwise write None.

Solution: Equations of vertical asymptotes: $x = 100$, $x = 0.2$

ii) Does the graph of $y = F(x)$ have any horizontal asymptotes? If so, write their equations, otherwise write None

Solution: $y = \frac{1}{10}$

iii) Find the zeros of the function $F(x)$. If the function does not have zeros, write None.

Solution: $x = 2$

6. [10 points]

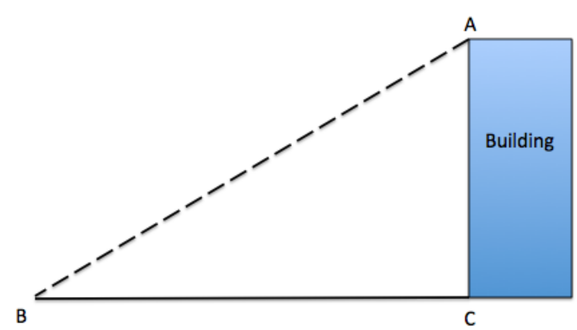
a. [5 points] The temperature T (in degrees Fahrenheit) at a point next to a campfire is inversely proportional to the square of its distance d (in meters) from the fire. If the temperature at a point 0.5 meters away from the fire is 500° F, what is the temperature (in degrees Fahrenheit) at 1.5 meters away from the fire? Show all your work to receive full credit.

Solution: $T = \dfrac{k}{d^2}$ so $500 = \dfrac{k}{0.5^2}$ and $k = 500(0.5)^2 = 125$.
Thus the temperature at 1.5 meters is $T = \dfrac{125}{1.5^2} \approx 55.56°$ F

b. [2 points] Let $H(x) = (x^3+1)^2$. Find two functions $K(x)$ and $J(x)$ such that $K(J(x)) = H(x)$. Your functions should satisfy $K(x) \neq x$ and $J(x) \neq x$.

Solution: $K(x) = x^2$ $\quad J(x) = x^3 + 1$ $\quad$ or $\quad K(x) = (x+1)^2$ $\quad J(x) = x^3$

c. [3 points] The shadow (the segment BC) made by a 150-foot-tall building has a length of 200 feet. Find the value, in **radians**, of the angle ABC.



Solution: Let θ = angle ABC, then $\tan\theta = \frac{150}{200}$. Hence $\tan^{-1}(\frac{150}{200}) \approx 0.643$ radians.

7. [10 points] The following table gives some values of the two functions $f(x)$ and $g(x)$.

x	2	8
$f(x)$	20	160
$g(x)$	4	12

a. [5 points] Suppose the function $f(x)$ is an exponential function. Find a formula for $f(x)$. Your answer must be in **exact form**. Show all your work.

Solution:

$$ab^8 = 160$$
$$ab^2 = 20$$
$$\frac{ab^8}{ab^2} = \frac{160}{20} \qquad b^6 = 8 \qquad \text{hence} \qquad b = 8^{\frac{1}{6}}$$
$$20 = a8^{\frac{2}{6}} = 2a \qquad \text{then} \qquad a = 10.$$
$$f(x) = 10\,(8)^{\frac{x}{6}}$$

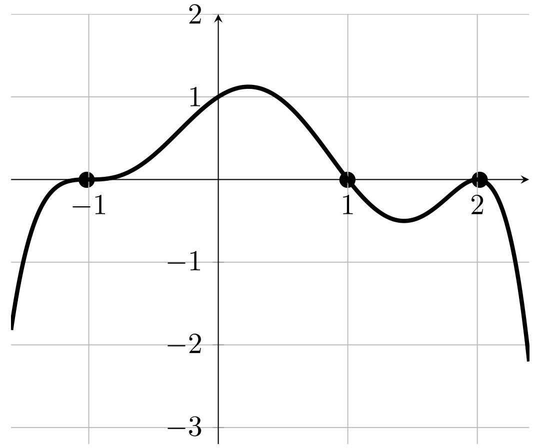
b. [5 points] Suppose that $g(x)$ is a power function. Find a formula for $g(x)$. Your answer must be in **exact form**. Show all your work.

Solution:

$$k(8^p) = 12$$
$$k(2^p) = 4$$
$$\frac{k8^p}{k2^p} = \frac{12}{4} = 3 \qquad \text{then} \qquad 4^p = 3.$$
$$\ln(4^p) = \ln(3) \qquad \text{implies} \qquad p = \frac{\ln(3)}{\ln(4)}$$
$$4 = k2^{\frac{\ln 3}{\ln 4}} \qquad \text{yields} \qquad k = \frac{4}{2^{\frac{\ln 3}{\ln 4}}}$$

$$g(x) = \frac{4}{2^{\frac{\ln 3}{\ln 4}}} x^{\frac{\ln 3}{\ln 4}}.$$

8. [8 points] The graph of the polynomial $p(x)$ is given below.



i) What are the zeros of the polynomial $p(x)$?

Answer: ______________________.

Solution: $x = -1, 1, 2$

ii) What is the vertical intercept of the graph of $p(x)$?

Answer: ______________________.

Solution: $p(0) = 1$ or $(0, 1)$

iii) Assume that the polynomial $p(x)$ has degree six. Use the vertical intercept to find a formula for $p(x)$.

$p(x) =$ ______________________.

Solution: Let $p(x) = a(x+1)^3(x-1)(x-2)^2$. Since $p(0) = 1$, we have that

$$1 = a(1)^3(-1)(-2)^2 = -4a.$$

So $p(x) = -\frac{1}{4}(x+1)^3(x-1)(x-2)^2$.

9. [12 points] Jemma and Sarah want to design a website for the winter sale of the store Fritz. The sale will start at 8 am and close at 8 pm on December 23. To build the website, they have to be able to predict the number of online customers that day. Each one has different predictions for the number of online customers that day.

a. [6 points] Sarah believes that the number of online customers will start at a minimum of 2 thousand online customers at 8 am and then it will increase to a maximum of 12 thousand customers at 2 pm. Let $S(t)$ be the sinusoidal function which gives the amount of online customers on the website (in thousands) *t hours after 8 am* on December 23 according to Sarah's predictions.

i) What are the amplitude and the midline of $S(t)$?

Solution: Amplitude=5 Midline: $y = 7$.

ii) Find a formula for the function $S(t)$ for $0 \leq t \leq 12$.

Solution: The amplitude is 5, the midline is $y = 7$ and the period is 12 hours. Since $S(t)$ has a minimum at $t = 0$, we get that

$$S(t) = -5\cos\left(\frac{\pi}{6}t\right) + 7$$

(*The statement of the problem has been included below for your own convenience*)
Jemma and Sarah want to design a website for the winter sale of the store Fritz. The sale will start at 8 am and close at 8 pm on December 23. To build the website, they have to be able to predict the number of online customers that day. Each one has different predictions for the number of online customers that day.

b. [6 points] On the other hand, Jemma believes that there will be 3 thousand online customers at 8 am. She expects that the number of online customers will reach a maximum of 10 thousands at 2 pm.
Let $J(t)$ be the quadratic function which gives the amount of online customers on the website (in thousands) *t hours after 8 am* on December 23 according to Jemma's predictions.

i) What is the vertex of $J(t)$?

Solution: The vertex is (6,10) since maximum is at 2 pm ($t = 6$) with 10 thousands online customers.

ii) Find a formula for $J(t)$ for $0 \le t \le 12$.

Solution: The vertex is (6,10). So the quadratic function J in vertex form is given by

$$J(t) = a(t-6)^2 + 10.$$

Since $J(0) = 3$, we have that $3 = a(-6)^2 + 10$ and $a = -\frac{7}{36}$. Thus

$$J(t) = -\frac{7}{36}(t-6)^2 + 10.$$

10. [10 points]

a. [5 points] Find all the values of $-4 \leq x \leq 20$ that satisfy the following equation. Find your answers algebraically. Your answer(s) must be in **exact form**. Show all your work.

$$2 - 6\sin\left(\frac{\pi}{8}x\right) = 4$$

Solution:

$$\begin{aligned} 2 - 6\sin\left(\frac{\pi}{8}x\right) &= 4 \\ \sin\left(\frac{\pi}{8}x\right) &= -\frac{1}{3} \\ \frac{\pi}{8}x &= \sin^{-1}\left(-\frac{1}{3}\right) \\ x_1 &= \frac{8}{\pi}\sin^{-1}\left(-\frac{1}{3}\right) \end{aligned}$$

$x_1 = x_1$
$x_2 = 8 - x_1$
$x_3 = 16 + x_1$.

b. [5 points] Let $w = F(s)$, where $F(s) = 4 + \ln(3^s + 1)$. Find a formula for $F^{-1}(s)$. Show all your work.

Solution:

$$\begin{aligned} w &= 4 + \ln(3^s + 1) \\ w - 4 &= \ln(3^s + 1) \\ e^{w-4} &= 3^s + 1 \\ e^{w-4} - 1 &= 3^s \\ \ln(e^{w-4} - 1) &= s\ln(3) \\ s &= \frac{\ln(e^{w-4} - 1)}{\ln(3)} = F^{-1}(w). \end{aligned}$$