Math 105 — First Midterm

October 10, 2016

UMID: Solutions   Section: ________

Instructor: ________

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1. **Do not open this exam until you are told to do so.**
2. **This exam has 10 pages (*not* including this cover page) and there are 9 problems in total.**
3. **Turn off and put away all cell phones, pagers, headphones, smartwatches and any other unauthorized electronic devices.**
4. Note that the problems are not of equal difficulty, so you may want to skip over a problem if you get stuck and return to it later.
5. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and notify your instructor when you hand in the exam.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. You must show an appropriate amount of work (including appropriate explanation) for each problem unless indicated otherwise: the graders will be looking not only for your answer, but also how you obtained it. Include units in your answer where appropriate.
8. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are not allowed a notecard for this exam.
9. If you use a graph that wasn't provided to you in answering a problem, be sure to include an explanation and sketch of the graph. If your solution to a problem makes reference to a table, it should be clear which entries of the table you're using.
10. You must solve the problems on this exam using the methods you have learned in this course.

Problem	**Points**	**Score**
1	13	
2	10	
3	13	
4	9	
5	11	
6	13	
7	10	
8	12	
9	9	
Total	100	

I RECOGNIZE THAT THERE ARE SEVERE PENALTIES FOR ACADEMIC DISHONESTY AND
I AFFIRM THAT I HAVE DONE NOTHING TO COMPROMISE THE INTEGRITY OF THIS EXAM,
NOR DO I INTEND TO HELP ANYONE ELSE DO SO.

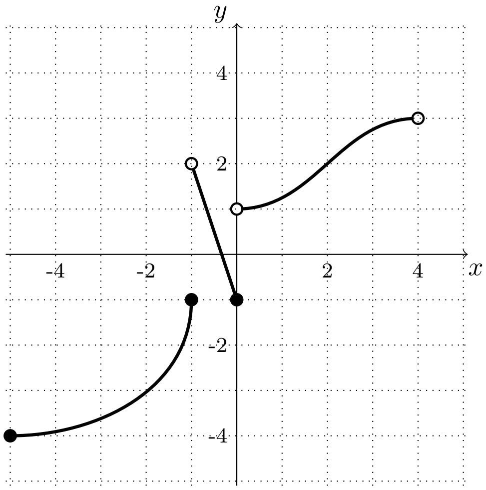
Initials: ________

1. [13 points] Let $j(x)$ be a function with domain $[-10, 13]$; some values of $j(x)$ are given in the table below.

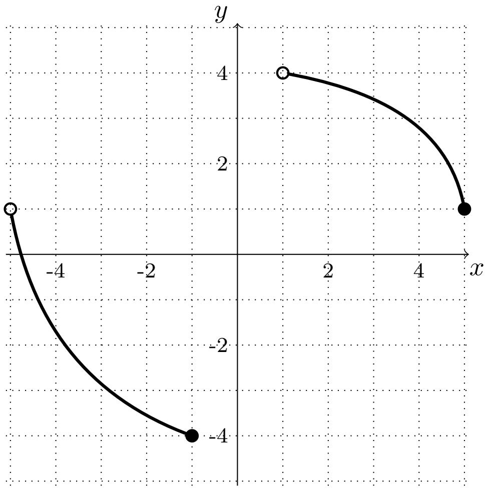
x	-10	-4	0	1	7	13
$j(x)$	-4	-2	1	1.5	1.8	1.9

The graphs of $y = k(x)$ and $y = \ell(x)$ are given below. Note that $\ell(x)$ is an invertible function.

GRAPH OF $y = k(x)$



GRAPH OF $y = \ell(x)$



a. [4 points] Based on the information above, which of the following statements *could* be true about the function $j(x)$ on $[-10, 13]$? **Circle all that apply**. You may use the space below for scratch work, but **you do not need to show any work for this part**.

$j(x)$ is concave up.

[✓ *$j(x)$ is an increasing function.*]

$j(x)$ is a linear function.

$j(x)$ is concave down.

$j(x)$ is a decreasing function.

$j(x)$ is a vertical shift of k.

[✓ *$j(x)$ is neither concave up nor concave down.*]

$j(x)$ is a quadratic function with vertex $(0, 1)$.

$j(x) = ab^x$ for some constants $a, b > 0$.

b. [2 points] What is the range of the function $\ell(x)$? **You do not need to show any work for this part**, but write your answer *in the space provided, using inequalities or interval notation.*

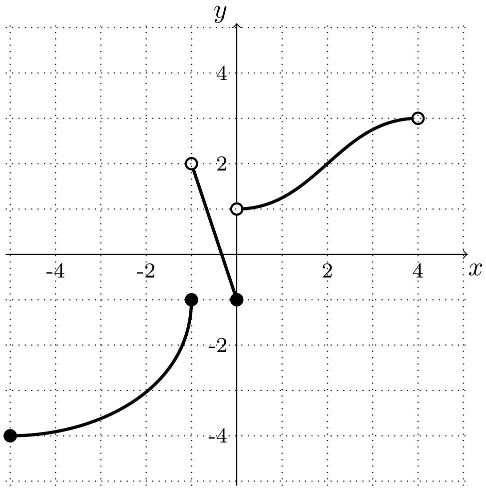
The range of $\ell(x)$ is $[-4, 4)$

THIS PROBLEM CONTINUES ON THE NEXT PAGE

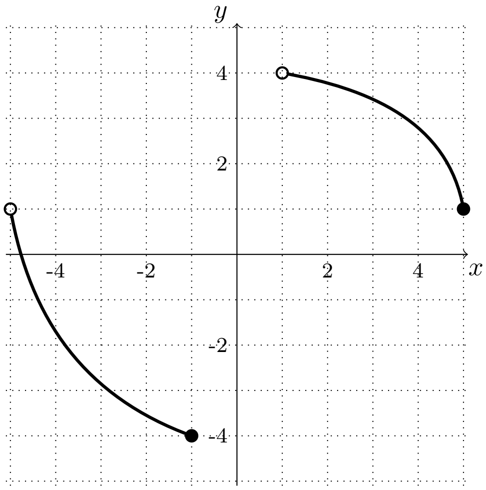
The table for $j(x)$, as well as the graphs of $y = k(x)$ and $y = \ell(x)$, have been reproduced below for your convenience.

x	-10	-4	0	1	7	13
$j(x)$	-4	-2	1	1.5	1.8	1.9

GRAPH OF $y = k(x)$



GRAPH OF $y = \ell(x)$



c. [4 points] Evaluate the following expressions, writing your answers in the space provided. If the expression cannot be evaluated based on the information given, write UNDEFINED. You may use the space below for scratch work, but **you do not need to show any work for this part**.

$\ell^{-1}(5)$ UNDEFINED

$j(-4)^{-1}$ $-\frac{1}{2}$

$\ell^{-1}(1)$ 5

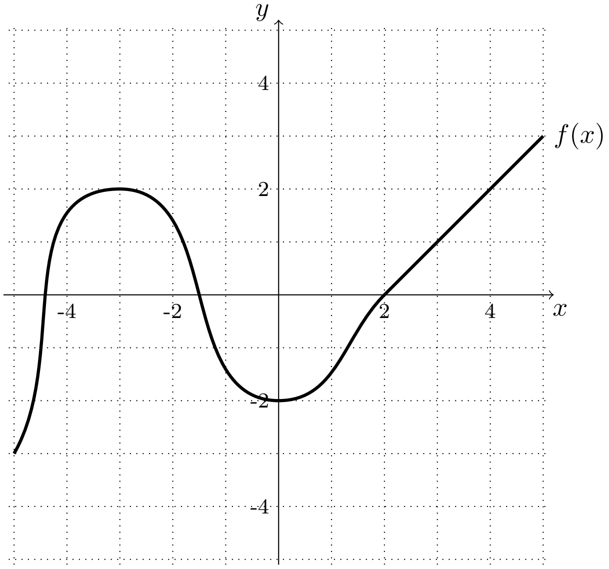
$j(k(-5))$ -2

d. [3 points] Find all values of x for which $\ell(k(x)) = -4$. **Show your work** and write your answer in the space provided. Write NONE if there are no such values of x.

Solution: From the graph of $y = \ell(x)$, we see that $\ell(k(x)) = -4$ when $k(x) = -1$. This happens when $x = -1$ or when $x = 0$.

$\ell(k(x))$ is -4 for $x = 0, -1$

2. [10 points] A portion of the graph of $y = f(x)$ is given below. **You do not need to show any work for this problem**.



a. [2 points] For which values of x must $f(x)$ be decreasing? Use only the information provided in the graph above, and write your answer *in the space provided, using inequalities or interval notation.*

$f(x)$ is decreasing on $[-3, 0]$

b. [2 points] Let $g(x) = f(x + 5) - 8$. For which values of x must $g(x)$ be decreasing? Use only the information provided in the graph above, and write your answer *in the space provided, using inequalities or interval notation.*

$g(x)$ is decreasing on $[-8, -5]$

c. [4 points] On which of the following intervals is the average rate of change of $f(x)$ the greatest? On which of the following intervals is it the least? Write your answers *in the spaces provided.* (Note: *greatest* and *least* do **not** mean largest and smallest in absolute value.)

$[-4, -1.5]$ $\quad$ $[-3, 0]$ $\quad$ $[-4, 4]$ $\quad$ $[2, 4]$ $\quad$ $[-5, 5]$

The average rate of change is the greatest on $[2, 4]$, *and the least on* $[-3, 0]$

d. [2 points] The line $y = 7$ is a horizontal asymptote for the graph of $y = f(x)$ (note that this is not shown in the graph above). Find the equation(s) of the horizontal asymptote(s) of the graph of $y = f(x - 10) + 4$ and write your answer(s) *in the space provided*, or **circle** THE GRAPH HAS NO HORIZONTAL ASYMPTOTES if appropriate.

Horizontal asymptote(s): $y = 11$

THE GRAPH HAS NO
HORIZONTAL ASYMPTOTES

3. [13 points] In part (a) of this problem, you should **show your work** and make sure your answers are **exact**. Note that *part (b) is independent of part (a).*

a. [9 points] There are $T(d)$ termites in an abandoned house on day d. Starting at $d = 0$, the population of termites increases by 30% each day, and reaches a peak of 28,561 termites at $d = 4$. Starting at $d = 4$, the termite population declines at a constant rate, up until $d = 8$ when there are no termites left. Write a *piecewise-defined* formula for $T(d)$ in terms of d *in the spaces provided.*

Solution: From the information above, we see that $T(d)$ is exponential on $0 \leq d \leq 4$ and linear on $4 < d \leq 8$.

For $0 \leq d \leq 4$: We know $T(d)$ is exponential with percentage growth rate 0.3, so $T(d) = a(1.3)^d$. To find a, we know that $T(4) = 28,561$, so $a(1.3)^4 = 28,561$. Dividing by 1.3^4 gives us $a = 10,000$.

For $4 < d \leq 8$: We know $T(d)$ is linear with average rate of change:

$$\frac{0 - 28,561}{4} = -7,140.25$$

Since $T(8) = 0$, using point-slope form gives us $T(d) = -7,140.25(d - 8)$.

$$T(d) = \begin{cases} \underline{\quad 10,000(1.3)^d \quad} & \text{if } \underline{\quad 0 \leq d \leq 4 \quad} \\ \underline{\quad -7,140.25(d-8) \quad} & \text{if } \underline{\quad 4 < d \leq 8 \quad} \end{cases}$$

b. [4 points] The termites at the abandoned house have begun attracting birds. The number of birds B, along with the temperature T (in °F) and the wind speed W (in miles per hour) have been recorded at various times h, where h is measured in hours after 8 a.m. on October 10.

h	0	1	2	3	4	5
B	10	11	15	13	11	5
T	30	33	40	39	33	31
W	14	10	13	12	11	10

Based on the table above, which of the following statments *could* be true about h, B, T and W? **Circle all that apply**.

[✓ *B is a function of T*]

[✓ *T is a function of B*]

W is a function of B

B is a function of W

h is a function of T

W is a function of T

4. [9 points] In both parts of this problem, you should **show your work** and write your final answers *in the spaces provided.* Note that *part (b) is independent of part (a).*

a. [4 points] When the brakes on a car are applied at full force while the car is moving, the car does not just come to an immediate halt. In fact, a car initially traveling at a speed v (measured in meters per second) will travel an additional $D(v)$ meters before coming to a complete stop, where

$$D(v) = v\left(2 + \frac{v}{14}\right) \qquad \text{for } v \geq 0$$

If it took a car 100 meters to come to a complete stop, how fast was it moving before the brakes were applied? Your final answer should be found *algebraically* and can be exact or accurate to three decimal places.

Solution: We need to find the values of v for which $D(v) = 100$. This happens when $\frac{1}{14}v^2 + 2v - 100 = 0$ which, by the quadratic formula, means:

$$v = \frac{-2 \pm \sqrt{4 + \frac{400}{14}}}{\frac{1}{7}}$$

so $v \approx 25.950$ or $v \approx -53.950$. Since we have $v \geq 0$ in the problem, this means that the car was traveling at approximately 25.950 meters per second.

The car was traveling at a speed of 25.950 m/s

b. [5 points] Martin is visiting the planet Nomae and throws a rock vertically upwards into the air. It takes the rock 0.5 seconds for it to reach its maximum height of 4 meters above the ground, and the rock was 1.5 meters above the ground when Martin released it. Find a formula for the height $h(t)$ (in meters) of the rock above the ground in terms of the time t (in seconds) elapsed since the rock was released, given that $h(t)$ is a quadratic function of t.

Solution: We know that $h(t)$ is a quadratic function with vertex $(0.5, 4)$, and hence $h(t) = a(t - 0.5)^2 + 4$. We also know that $h(0) = 1.5$, so $a(-0.5)^2 + 4 = 1.5$, and hence $a = \frac{-2.5}{0.25} = -10$.

$h(t) =$ $-10(t - 0.5)^2 + 4$

5. [11 points] In both parts of this problem, you should **show your work**, and make sure your answers are **exact** and written *in the spaces provided.*

a. [6 points] Kayla was cultivating a strain of bacteria in her lab, and noticed that the mass of her bacterial culture was growing exponentially. She started the experiment at 9 a.m. and ended it at 5 p.m., at which point she had 234 grams of bacteria. Find a formula expressing the mass of her culture $m(t)$ (in grams) as a function of the time t, measured in hours after 9 a.m., given that the mass of her culture was 20 grams at noon.

Solution: Since $m(t)$ is exponential, our formula is of the form $m(t) = ab^t$ for some constants a and b. We know that $m(3) = 20$ and $m(8) = 234$, which gives us:

$$ab^8 = 234$$
$$ab^3 = 20$$

and dividing the first equation by the second gives us:

$$b^5 = 11.7$$

so $b = 11.7^{\frac{1}{5}}$. To get a, we plug this back into the second equation, which gives us:

$$a\left(11.7^{\frac{1}{5}}\right)^3 = 20$$

and so $a = 20 \cdot 11.7^{-\frac{3}{5}}$.

$m(t) =$ $20 \cdot 11.7^{-\frac{3}{5}} \cdot (11.7^{\frac{1}{5}})^t$

b. [5 points] A 10 liter bottle is filled completely with a combination of oil and vinegar. Each kilogram of oil takes up 1.25 liters, while each kilogram of vinegar takes up 1 liter. Let $N(\ell)$ be the amount of vinegar (measured in kilograms) in the bottle when it is filled with ℓ kilograms of oil. Find a formula for $N(\ell)$ in terms of ℓ and indicate the domain on which your formula is valid. *Note: there are practical considerations for your domain in this problem.*

Solution: Since the oil and vinegar fill up the bottle completely, we have $(1.25 \cdot \ell) + (1 \cdot N(\ell)) = 10$, and so $N(\ell) = 10 - 1.25 \cdot \ell$. Of course, ℓ cannot be negative, and the largest ℓ can be is when the bottle is filled with oil and has no vinegar. In other words, ℓ is the largest when $1.25\ell = 10$ and therefore $\ell = 8$. So the domain of $N(\ell)$ is $0 \leq \ell \leq 8$.

$N(\ell) =$ $10 - 1.25 \cdot \ell$, with domain $0 \leq \ell \leq 8$

6. [13 points] Jared, Katia and Rory run together frequently on their university's track. The number of calories Jared, Katia and Rory each burn in running n laps is given by the functions $J(n)$, $K(n)$ and $R(n)$ respectively. You may assume that all of these functions are invertible.

a. [2 points] Jared always burns exactly as many calories as Katia does when he runs 5 more laps than she does. Write an expression for $J(n)$ involving the function K that represents this fact.

Solution: To burn the same number of calories as Jared does when he runs n laps, Katia needs to only run $n - 5$ laps. So we have $J(n) = K(n - 5)$.

$J(n) =$ $K(n - 5)$

b. [2 points] Rory has recently added a 10 minute bike ride to his workout routine, during which he burns an additional 150 calories. Let $H(n)$ be the total number of calories that Rory burns (both running and biking) if he runs n laps and bikes for 10 minutes afterwards; write an expression for $H(n)$ involving the function R that represents this fact.

Solution: If Rory runs n laps and bikes for 10 minutes, he burns $R(n) + 150$ calories.

$H(n) =$ $R(n) + 150$

c. [3 points] Interpret $J^{-1}(100)$ in practical terms.

Solution: $J^{-1}(100)$ is the number of laps Jared must run to burn 100 calories.

Like most people, Jared bases the length of his run on the average temperature forecast for that day: he decides to run $G(m)$ laps when the projected average temperature is m (in degrees Fahrenheit).

d. [3 points] Interpret $J(G(90))$ in practical terms.

Solution: $J(G(90))$ is the number of calories Jared burns when the temperature is 90 degrees Fahrenheit.

e. [3 points] The graph of the function $y = G(m)$ has a vertical intercept. Interpret this vertical intercept in practical terms.

Solution: The vertical intercept is the number of laps Jared runs when the temperature is 0 degrees Fahrenheit.

7. [10 points] Emanuel grows corn on his farm to produce corn oil, and is hoping to increase his income by producing more corn oil. Unfortunately, Emanuel has to be careful about exactly how many corn plants he grows: if he plants too much corn, he risks crowding the plants and causing some of them to wilt and die. However, he knows that if he plants an additional p tons of corn, he can produce a total of $C(p)$ gallons of corn oil, where $C(p)$ is given by the formula:

$$C(p) = -3p^2 + \sqrt{11}p + 200.$$

a. [3 points] Find the average rate of change of $C(p)$ on the interval $1 \leq p \leq 5$. Your answer can be exact or accurate to three decimal places, but should **include units**.

Solution: The average rate of change is

$$\begin{aligned}\frac{C(5) - C(1)}{5 - 1} &= \frac{-72 + 4\sqrt{11}}{4} \\ &\approx -14.683\end{aligned}$$

The average rate of change is -14.683 gallons/ton

b. [5 points] Write $C(p)$ in vertex form by completing the square. Your answer must be **exact**, and you must **show all your work**, **step-by-step**, to get full credit.

Solution: By completing the square, we get:

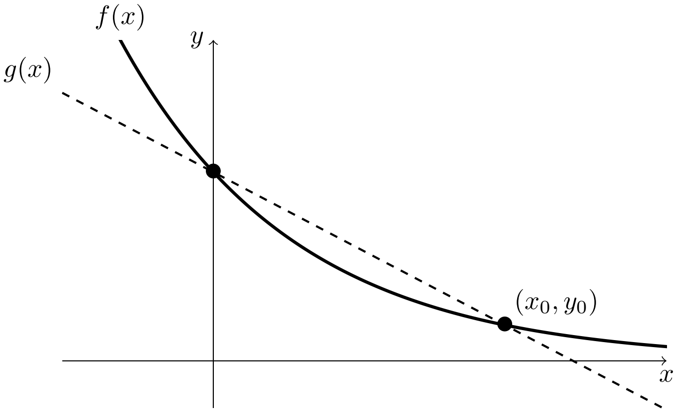
$$\begin{aligned}C(p) &= -3p^2 + \sqrt{11}p + 200 \\ &= -3\left(p^2 - \frac{\sqrt{11}}{3}p\right) + 200 \\ &= -3\left(p^2 - \frac{\sqrt{11}}{3}p + \frac{11}{36} - \frac{11}{36}\right) + 200 \\ &= -3\left(p^2 - \frac{\sqrt{11}}{3}p + \frac{11}{36}\right) + \frac{11}{12} + 200 \\ &= -3\left(p - \frac{\sqrt{11}}{6}\right)^2 + \left(200 + \frac{11}{12}\right)\end{aligned}$$

$C(p) =$ $-3\left(p - \frac{\sqrt{11}}{6}\right)^2 + \left(200 + \frac{11}{12}\right)$

c. [2 points] Based on your answer above, how much corn should Emanuel add to maximize his corn oil production? Your answer should be **exact** and **include units**.

Emanuel should add $\frac{\sqrt{11}}{6}$ tons of corn

8. [12 points] Let $f(x) = a^x$ and $g(x) = c + dx$ where a, c and d are constants. The graph of $y = f(x)$ and $y = g(x)$ are shown below. The point of intersection not lying on the y-axis has coordinates (x_0, y_0).



a. [10 points] In each of the bullet points below, you are asked to **circle** the option that must be true based on the graph above. If there is not enough information to decide on any of the options in a given row, circle NOT ENOUGH INFORMATION.

- The constants a and c satisfy:

  $a > c$ [✓ $a < c$] $a = c$ NOT ENOUGH INFORMATION

- The constants a and d satisfy:

  [✓ $a > d$] $a < d$ $a = d$ NOT ENOUGH INFORMATION

- The constants c and d satisfy:

  [✓ $c > d$] $c < d$ $c = d$ NOT ENOUGH INFORMATION

- If the constants a and c remain the same while the value of the constant d increases, then the value of x_0, the x-coordinate of the point of intersection of $f(x) = a^x$ and $g(x) = c + dx$:

  [✓ INCREASES] DECREASES STAYS THE SAME NOT ENOUGH INFORMATION

- If the constants a and c remain the same while the value of the constant d increases, then the value of y_0, the y-coordinate of the point of intersection of $f(x) = a^x$ and $g(x) = c + dx$:

  INCREASES [✓ DECREASES] STAYS THE SAME NOT ENOUGH INFORMATION

b. [2 points] The graph of the function $h(x)$ has a vertical intercept at $(0, -2)$ and is perpendicular to the graph of $g(x) = c + dx$. Find a formula for the function $h(x)$. Your formula may include any or all of the constants a, c and d.

Solution: The slope of $h(x)$ must be $-\frac{1}{d}$. Since it has vertical intercept -2, its equation must therefore be $h(x) = -\frac{1}{d}x - 2$.

$h(x) =$ $-\dfrac{1}{d}x - 2$

9. [9 points] On the grid below, sketch the graph of **a single function** $f(x)$ that fulfills all the requirements below.

- The domain of f is $[-6, 5)$
- The range of f is $[-5, 6]$
- $f(-1) = 2$ and the average rate of change of $f(x)$ on the interval $[-1, 3]$ is -1
- $f(x) < 1$ for $x < -1$
- f has a constant rate of change on the interval $[-6, -1)$.
- f is increasing on the interval $[-6, -1)$
- f is concave down on the interval $[-1, 1)$
- f is decreasing on the interval $[1, 3]$
- f is concave up on the interval $(3, 5)$