Math 105 — Second Midterm

November 14, 2016

UMID: Solutions ______________________ Section: ____________

Instructor: ______________________

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1. **Do not open this exam until you are told to do so**.
2. **This exam has 9 pages (*not* including this cover page) and there are 9 problems in total**.
3. **Turn off and put away all cell phones, pagers, headphones, smartwatches and any other unauthorized electronic devices**.
4. Note that the problems are not of equal difficulty, so you may want to skip over a problem if you get stuck and return to it later.
5. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and notify your instructor when you hand in the exam.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. You must show an appropriate amount of work (including appropriate explanation) for each problem unless indicated otherwise: the graders will be looking not only for your answer, but also how you obtained it. Include units in your answer where appropriate.
8. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are not allowed a notecard for this exam.
9. If you use a graph that wasn't provided to you in answering a problem, be sure to include an explanation and sketch of the graph. If your solution to a problem makes reference to a table, it should be clear which entries of the table you're using.
10. You must solve the problems on this exam using the methods you have learned in this course.

Problem	**Points**	**Score**
1	20	
2	11	
3	14	
4	11	
5	9	
6	6	
7	11	
8	10	
9	8	
Total	100	

I RECOGNIZE THAT THERE ARE SEVERE PENALTIES FOR ACADEMIC DISHONESTY AND
I AFFIRM THAT I HAVE DONE NOTHING TO COMPROMISE THE INTEGRITY OF THIS EXAM,
NOR DO I INTEND TO HELP ANYONE ELSE DO SO.

Initials: ____________

1. [20 points] **You do not need to show any work for this problem**, but you should write your answers *in the spaces provided.*

a. [2 points] Let $j(x)$ be an odd function with domain $(-\infty, \infty)$ and $j(-1) = 4$. Evaluate $j(0)$ and $j(1)$.

$j(0) =$ 0, *and* $j(1) =$ -4

b. [2 points] Let $f(x) = \log x$. Write down an expression for a function $g(x)$ which is a transformation of $f(x)$ that has a vertical asymptote at $x = -2$.

$g(x) =$ $\log(x + 2)$

c. [3 points] Let $f(x) = \log x$ and $h(x) = \log(0.5x)$. By how much, and in which direction, must the graph of $y = f(x)$ be shifted *vertically* to obtain the graph of $y = h(x)$? Your answer must be **exact**.

The graph of $y = f(x)$ *must be shifted vertically* down *by* $-\log(0.5)$

d. [4 points] Let $k(x) = b\sin(x) - 10$ (for some constant b) be a periodic function with amplitude 4. List *all* possible values of b, and find the equation of the midline of $y = k(x)$.

The midline is $y = -10$, *and* b *could be* 4 or -4

e. [3 points] Consider the graph of $y = \tan(x + 1)$. Write down the equations of *one* horizontal asymptote and *one* vertical asymptote of this graph, or write NONE if there are no asymptotes of a particular type. Your answer must be **exact**.

A vertical asymptote is $x = 0.5\pi - 1$, *and a horizontal asymptote is* NONE

f. [6 points] Let $R(x) = 2L(7x - 3) + 4$. List the transformations you need to apply to the graph of $y = L(x)$, in order, to obtain the graph of $y = R(x)$. Fill each space with either a *number* **or** *one of the phrases below*, as appropriate.

SHIFT IT HORIZONTALLY TO THE RIGHT	SHIFT IT HORIZONTALLY TO THE LEFT	SHIFT IT VERTICALLY UPWARDS	SHIFT IT VERTICALLY DOWNWARDS
COMPRESS IT HORIZONTALLY	STRETCH IT HORIZONTALLY	COMPRESS IT VERTICALLY	STRETCH IT VERTICALLY

To get the graph of $y = R(x)$, *we start with the graph of* $y = L(x)$.

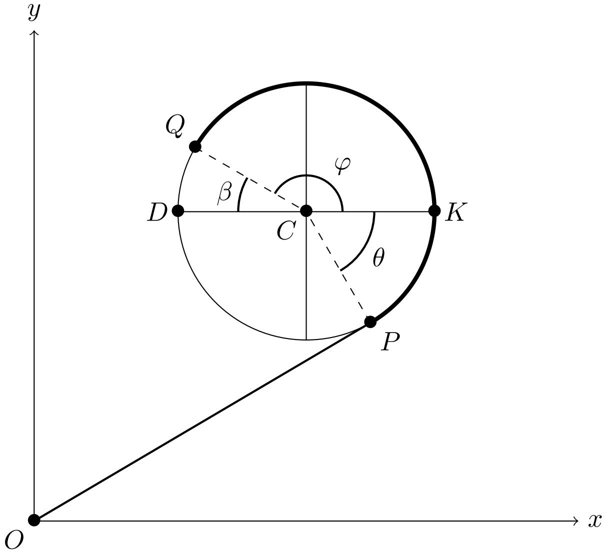
First*, we* COMPRESS IT HORIZONTALLY *by* 1/7,

and then *we* SHIFT IT HORIZONTALLY TO THE RIGHT *by* 3/7,

and then *we* STRETCH IT VERTICALLY *by* 2,

and then *we* SHIFT IT VERTICALLY UPWARDS *by* 4

2. [11 points] A drone starts at the origin O, and flies in a straight line to a point P with coordinates (a, b). From there, it travels counterclockwise around a circle of radius 8 centered at the point $C = (20, 15)$, until it reaches the point Q. This is illustrated in the diagram below, though **the diagram is *not* drawn to scale**.



Note that θ, β and φ are the *positive* measures of the angles PCK, DCQ and QCK (respectively) given in *radians*. **You do not need to show any work for this problem**, but you should write your answers *in the spaces provided*.

a. [2 points] Find the length of the line segment OP in terms of a and b *alone*.

The length of OP is $\sqrt{a^2+b^2}$

b. [2 points] Find a formula for φ in terms of β *alone*.

$\varphi =$ $\pi - \beta$

c. [3 points] Find the length of the (bolded) circular arc PQ in terms of θ and β *alone*.

The length of the circular arc PQ is $8(\theta + (\pi - \beta))$

d. [4 points] Write a formula for b in terms of θ *alone*.

$b =$ $15 - 8\sin\theta$

3. [14 points] At a wildlife sanctuary in central Africa, conservationists are carefully monitoring the population of various species of animals. For the following parts, write your answers *in the spaces provided.* Your answers for this problem can either be exact, or accurate to three decimal places.

a. [3 points] On January 1, 2008, the population of lions in the sanctuary was estimated to be 850, and was decreasing exponentially at a continuous rate of 25% each year. Find a formula for the population $L(t)$ of lions in the sanctuary t years after January 1, 2008. **You do not need to show any work for this part**.

$L(t) =$ $850e^{-0.25t}$

b. [5 points] On the other hand, the number of elephants in the sanctuary increased by 60% every 7 years. Let $E(t)$ be the number of elephants in the sanctuary t years after January 1, 2008. What is the (annual) continuous growth rate of the function E? You should carefully **show your work** for this part.

Solution: The function $E(t)$ is exponential, so we have $E(t) = ae^{kt}$ for some constants a and k. We know that $E(7) = 1.6a$, so:

$$\begin{aligned} 1.6a &= ae^{7k} \\ 1.6 &= e^{7k} \\ 7k &= \ln 1.6 \\ k &= \frac{1}{7}\ln 1.6 \end{aligned}$$

The continuous growth rate of E is $\frac{\ln(1.6)}{7}$ *per year.*

For the following parts, **you do not *need* to show any work**, but **you can receive partial credit for work shown if your final answer is incorrect**.

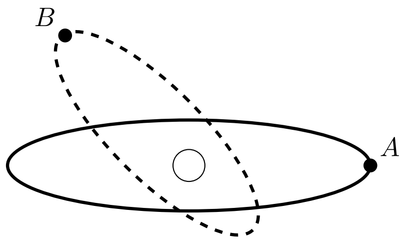
c. [3 points] Let $B(m) = 60(3)^{0.5m-1}$ be the number of buffalo in the sanctuary m months after July 15, 2016. What is the (monthly) continuous growth rate of the function B?

The continuous growth rate of B is $\ln(3^{0.5})$ *per month.*

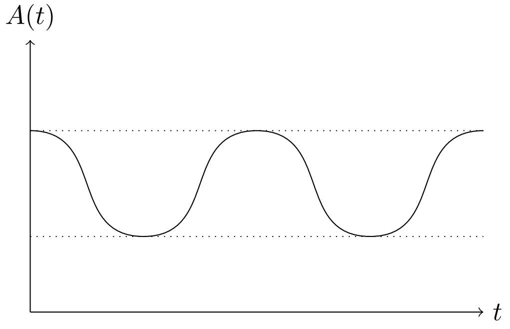
d. [3 points] Let $H(y)$ be the total value of donations received by the sanctuary's governing organization (in thousands of dollars) y years after July 15, 2016. The function H is exponential, with continuous growth rate $e^{0.77}$. What is the annual percentage growth rate of the function H?

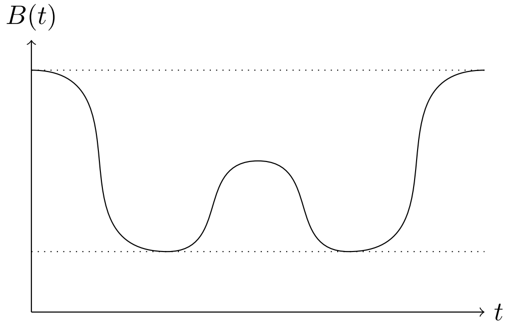
The annual percentage growth rate of H is $e^{(e^{0.77})} - 1$

4. [11 points] The comets A and B each orbit the sun in an ellipse, as illustrated in the diagram below. Let $A(t)$ be the distance (in millions of miles) between comet A and the sun and $B(t)$ the distance (in millions of miles) between comet B and the sun t years after June 12, 2013.



It takes 7 years for comet A to complete a full orbit (i.e. return to its initial position), and 20 years for comet B to do the same. The functions $A(t)$ and $B(t)$ have been graphed *for the time it takes the comets to travel through a complete orbit and return to their starting positions.*





Note that **the graphs are *not* drawn to the same scale. You do not need to show any work for this problem**.

a. [2 points] What is the period of the function $A(t)$? Write your answer *in the space provided.*

The period of $A(t)$ is 3.5 years

b. [2 points] What is the period of the function $B(t)$? Write your answer *in the space provided.*

The period of $B(t)$ is 20 years

c. [3 points] The closest that comet A gets to the sun is 1.75 million miles, and the function $A(t)$ has midline $y = 4.25$. What is the furthest that comet A gets from the sun? Write your answer *in the space provided*, and **include units**.

The furthest that comet A gets is 6.75 million miles.

d. [4 points] Comet C (not shown above) also orbits the sun in an ellipse. Let $C(t)$ be the distance (in millions of miles) from comet C to the sun t years after June 12, 2013.

The function $C(t)$ is periodic with period 4. Between $t = 0$ and $t = 4$, comet C is the closest to the sun at time $t = 3$. Which of the following **must** be true? **Circle** your answer(s) from the options listed; if none of the options are correct, circle NONE OF THESE.

$C(27) = C(32)$

$C(3) > C(4)$

$C(t)$ is the largest at $t = 1$

[✓ $C(28) = C(32)$]

[✓ $C(11) \leq C(2)$]

NONE OF THESE

5. [9 points] At low temperatures, pure tin can deteriorate and become brittle. Martín was recently awarded a tin medal, which he received in the mail; unfortunately, it had already begun to degrade by the time he had received it.

Let $P(t)$ be the fraction of the medal that has degraded t days after Martín first opened the package containing the medal, which is given by the formula:

$$P(t) = \frac{1}{1 + 4(2^{-kt})}$$

where $k > 0$ is a constant.

a. [2 points] What fraction of the medal had already degraded at the time Martín first opened the package? **Circle** your answer from the options below. If none of the options are correct, circle NONE OF THESE.

1 [✓ $\frac{1}{5}$] $\frac{1}{9}$ NONE OF THESE

b. [2 points] Let $U(t)$ be the fraction of the medal that has not yet degraded t days after Martín first opened the package. Find a formula that expresses $U(t)$ as a (combination of) transformation(s) of $P(t)$. **You do not need to show any work for this part**, but you should write your answer *in the space provided.*

$U(t) =$ $1 - P(t)$

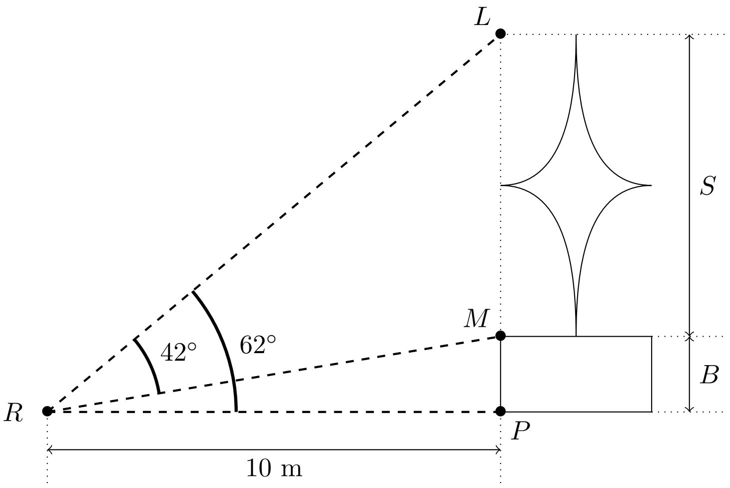
c. [5 points] Exactly 4 days after first opening the package, Martín finds that the fraction of the metal that has degraded is now 0.95. Set up an equation representing this fact, and use this to find k. Your answer must be found *algebraically*, and must be **exact**. You should carefully **show your work** for this part, and write your answer *in the space provided.*

Solution: We have $P(4) = 0.95$, which gives us:

$$\begin{aligned}
0.95 &= \frac{1}{1 + 4(2^{-4k})} \\
1 + 4(2^{-4k}) &= \frac{1}{0.95} \\
4(2^{-4k}) &= \frac{1}{0.95} - 1 \\
2^{-4k} &= \frac{1}{4}\left(\frac{1}{0.95} - 1\right) \\
-4k \ln 2 &= \ln\left(\frac{1}{4}\left(\frac{1}{0.95} - 1\right)\right) \\
k &= -\frac{1}{4\ln 2}\ln\left(\frac{1}{4}\left(\frac{1}{0.95} - 1\right)\right)
\end{aligned}$$

$k =$ $-\frac{1}{4\ln 2}\ln\left(\frac{1}{4}\left(\frac{1}{0.95} - 1\right)\right)$

6. [6 points] A sculpture of a star of height S (measured in meters) is mounted on a base of height B (measured in meters). Reina is standing at a distance of 10 meters away from the base of the statue, at the point R.



The measures of the angles LRM and LRP are 42° and 62° respectively. Note that **the diagram above is *not* drawn to scale**. For this problem, you should **show your work**, and write your answers *in the spaces provided.*

a. [3 points] Write an expression for the height S of the star. Your answer may involve the constant B.

Solution: Since LPR is a right angle, we have:

$$\tan(62^\circ) = \frac{S+B}{10}$$

and solving for S gives us:

$$S = 10\tan(62^\circ) - B$$

$S =$ $10\tan(62^\circ) - B$

b. [3 points] Find the height B of the base of the statue. Your answer for this part **cannot** involve S.

Solution: Since MPR is a right angle and the measure of angle MRP is $62^\circ - 42^\circ = 20^\circ$, we have:

$$\tan(20^\circ) = \frac{B}{10}$$

and solving for B gives us:

$$B = 10\tan(20^\circ)$$

$B =$ $10\tan(20^\circ)$

7. [11 points] In each of the following parts, you are given an equation in which you must solve for x. Your answers must be **exact** and should be obtained *algebraically*. You should **show all your work**, **step-by-step**, and write your final answers *in the spaces provided*.

a. [3 points] $\ln(3x^7+5)=-2$

Solution: We exponentiate both sides and solve for x, which gives us:

$$\begin{aligned} 3x^7+5 &= e^{-2} \\ 3x^7 &= e^{-2}-5 \\ x^7 &= \frac{1}{3}\left(e^{-2}-5\right) \\ x &= \sqrt[7]{\frac{1}{3}\left(e^{-2}-5\right)} \end{aligned}$$

$x =$ $\sqrt[7]{\frac{1}{3}\left(e^{-2}-5\right)}$

b. [4 points] $e^{7x}=5e^{10x}$

Solution: We take the natural logarithm of both sides and use properties of the logarithm to simplify, which gives us:

$$\begin{aligned} \ln\left(e^{7x}\right) &= \ln\left(5e^{10x}\right) \\ 7x &= (\ln 5)+10x \\ -3x &= \ln 5 \\ x &= -\frac{1}{3}\ln 5 \end{aligned}$$

$x =$ $-\frac{\ln 5}{3}$

c. [4 points] $4(\log(ax))^3+8=0$, where $a>0$ is a constant. Your answer for this part may involve a.

Solution: We'll first isolate the $(\log(ax))^3$ on one side, then take a cube root and exponentiate to solve for x:

$$\begin{aligned} (\log(ax))^3 &= -2 \\ \log(ax) &= \sqrt[3]{-2} \\ ax &= 10^{\sqrt[3]{-2}} \\ x &= \frac{1}{a}10^{\sqrt[3]{-2}} \end{aligned}$$

$x =$ $\frac{1}{a}10^{\sqrt[3]{-2}}$

8. [10 points] Twissell is attempting to determine the depth of a lake at various points by lowering a sensor to the bottom of the lake and measuring the intensity of the sun's light at that point. According to his calculations, if the intensity of the light (in lumens) that reaches the sensor is I, then the sensor must be at a depth of $F(I)$ (measured in meters), which is given by the formula:

$$F(I) = -\frac{1}{4}\log\left(\frac{I}{c}\right)$$

where $c > 0$ is a constant. Note that the intensity of light I at all points underwater is positive and smaller than c, so $F(I)$ is positive.

In the following parts, you must **show all your work**, **step-by-step**, and find your answers *algebraically* to receive full credit. Your final answers must be **exact**, and should be written *in the spaces provided.*

a. [4 points] What is the intensity of the light that reaches the sensor when it is 2 meters underwater? Your answer for this part may include the constant c, and should **include units**.

Solution: Two meters below the surface, the intensity I should satisfy:

$$\begin{aligned} 2 &= -\frac{1}{4}\log\left(\frac{I}{c}\right) \\ \log\left(\frac{I}{c}\right) &= -8 \\ \frac{I}{c} &= 10^{-8} \\ I &= c\cdot 10^{-8} \end{aligned}$$

The intensity 2 meters below the surface is $c\cdot 10^{-8}$

b. [6 points] Twissell submerges the sensor at two different points in the lake.

- At the first point, the depth is d meters and the sensor measures the intensity of the sun's light to be $6K$ lumens.
- At the second point, the depth is D meters and the sensor measures the intensity of the sun's light to be K lumens.

How much deeper is the lake at the second point compared to the first point? Your final answer should be simplified so that it does **not** include the constants K *or* c, but should **include units**.

Solution: We know that:

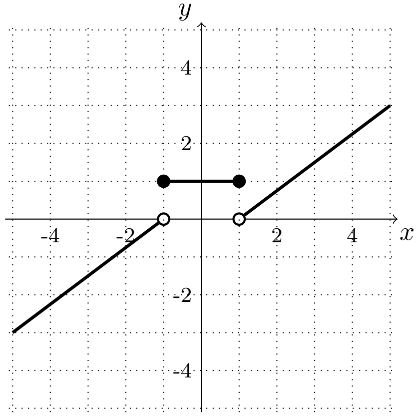
$$\begin{aligned} D - d &= -\frac{1}{4}\log\left(\frac{K}{c}\right) + \frac{1}{4}\log\left(\frac{6K}{c}\right) \\ &= -\frac{1}{4}\left(\log\left(\frac{K}{c}\right) - \log\left(\frac{6K}{c}\right)\right) \end{aligned}$$

which we can simplify using properties of the logarithm to get:

$$\begin{aligned} &= -\frac{1}{4}\left(\log\left(\frac{K}{c}\cdot\frac{c}{6K}\right)\right) \\ &= -\frac{1}{4}\log\left(\frac{1}{6}\right) \end{aligned}$$

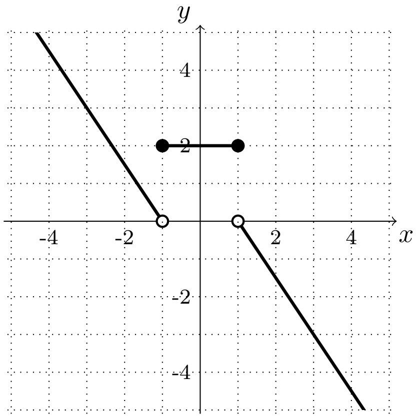
$D - d =$ $-\frac{1}{4}\log\left(\frac{1}{6}\right)$

9. [8 points] Consider the graph of $y = f(x)$ given below.

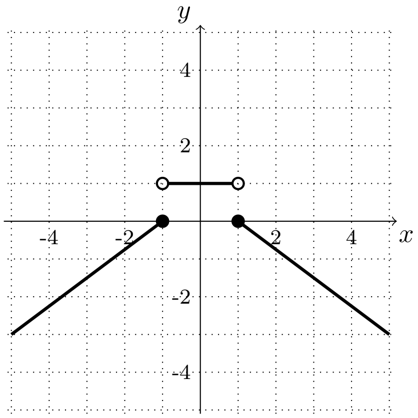


For each of the graphs below, if the graph is a (combination of) transformation(s) of the graph of $y = f(x)$, write an expression *in the space provided* that gives this (combination of) transformation(s). If the given graph is not a combination of vertical and horizontal shifts, stretches, compressions and reflections of the graph of $y = f(x)$, write NOT A TRANSFORMATION.

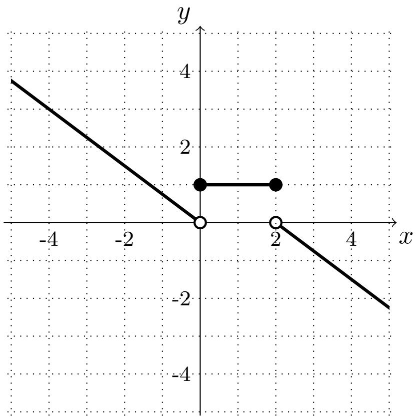
You do not need to show any work for this problem.



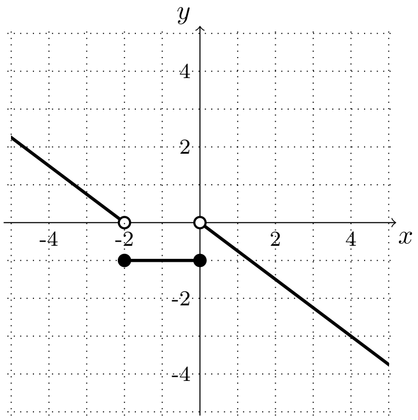
This is the graph of $y =$ $2f(-x)$



This is the graph of $y =$ NOT A TRANSFORMATION



This is the graph of $y =$ $f(-x+1)$



This is the graph of $y =$ $-f(x+1)$