Math 105 — Final Exam

December 19, 2016

UMID: Solutions   Section: ________

Instructor: ____________________

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1. **Do not open this exam until you are told to do so**.
2. **This exam has 13 pages (*not* including this cover page) and there are 11 problems in total**.
3. **Turn off and put away all cell phones, pagers, headphones, smartwatches and any other unauthorized electronic devices**.
4. Note that the problems are not of equal difficulty, so you may want to skip over a problem if you get stuck and return to it later.
5. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and notify your instructor when you hand in the exam.
6. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. You must show an appropriate amount of work (including appropriate explanation) for each problem unless indicated otherwise: the graders will be looking not only for your answer, but also how you obtained it. Include units in your answer where appropriate.
8. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course. You are not allowed a notecard for this exam.
9. If you use a graph that wasn't provided to you in answering a problem, be sure to include an explanation and sketch of the graph. If your solution to a problem makes reference to a table, it should be clear which entries of the table you're using.
10. You must solve the problems on this exam using the methods you have learned in this course.

Problem	**Points**	**Score**
1	8	
2	7	
3	8	
4	8	
5	12	
6	11	
7	8	
8	15	
9	7	
10	11	
11	5	
Total	100	

I RECOGNIZE THAT THERE ARE SEVERE PENALTIES FOR ACADEMIC DISHONESTY AND I AFFIRM THAT I HAVE DONE NOTHING TO COMPROMISE THE INTEGRITY OF THIS EXAM, NOR DO I INTEND TO HELP ANYONE ELSE DO SO.

Initials: ________

1. [8 points] Radioactive waste has been draining into a lake, causing trees in the surrounding forest to wilt and die. On March 24, 2010, local scientists surveyed the surrounding forest, determining that $S(m)$ thousand trees had begun to wilt m kilometers out from the center of the lake. The scientists determined that 20 thousand trees were decaying 20 kilometers out, and 5 thousand trees were decaying 60 kilometers out.

a. [3 points] Assume that the function S is invertible. Find the average rate of change of the inverse function S^{-1} on the interval $[5, 20]$. Write your final answer *in the space provided*, and **include units**.

Solution: The average rate of change is

$$\frac{S^{-1}(20) - S^{-1}(5)}{20-5} = \frac{20-60}{20-5} = -\frac{40}{15}$$

The average rate of change of S^{-1} *on the interval* $[5,20]$ *is* $-\frac{40}{15}$ kilometers/thousand trees

b. [5 points] Find a formula for $S(m)$ in terms of m, assuming that S is a power function of m. Your answer should be **exact**, and you must **show your work** carefully, writing your final answer *in the space provided.*

Solution: If $S(m)$ is a power function, it must have formula am^b for some constants a, b. We know that $S(20) = 20$ and $S(60) = 5$. This gives us:

$$a(20)^b = 20$$
$$a(60)^b = 5$$

If we divide the first equation by the second, we get:

$$\frac{(20)^b}{(60)^b} = \frac{20}{5}$$

which simplifies to:

$$\left(\frac{1}{3}\right)^b = 4$$

Taking the natural logarithm of both sides of this equation and simplifying gives us:

$$b \ln\left(\frac{1}{3}\right) = \ln 4$$
$$b = \frac{\ln 4}{\ln\left(\frac{1}{3}\right)}$$
$$b = -\frac{\ln 4}{\ln 3}$$

And, finally, plugging this into the first equation gives us:

$$a(20)^{-(\ln 4)/(\ln 3)} = 20$$
$$a = \frac{20}{20^{-(\ln 4)/(\ln 3)}}$$

$S(m) =$ $\frac{20}{20^{-(\ln 4)/(\ln 3)}} m^{-(\ln 4)/(\ln 3)}$

2. [7 points] Olga runs a factory that produces pitch, and finds that the cost C (in thousands of dollars) to produce g gallons of pitch is given by $C = f(g)$, where:

$$f(g) = 5 + \log\left(3 + e^{7g}\right)$$

for $g \geq 0$. Note that f is an invertible function.

a. [5 points] Find a formula for the quantity of pitch $f^{-1}(C)$ (in gallons) that the factory must have produced in terms of the total cost C (in thousands of dollars) incurred. You must **show your work** carefully for this part.

Solution: We solve for g in the equation given, first subtracting 5:

$$\begin{aligned} C &= 5 + \log\left(3 + e^{7g}\right) \\ \log\left(3 + e^{7g}\right) &= C - 5 \end{aligned}$$

We exponentiate both sides to remove the natural logarithm:

$$\begin{aligned} 3 + e^{7g} &= 10^{(C-5)} \\ e^{7g} &= 10^{(C-5)} - 3 \end{aligned}$$

And we now take a natural logarithm to isolate g:

$$\begin{aligned} 7g &= \ln\left(10^{(C-5)} - 3\right) \\ g &= \frac{1}{7}\ln\left(10^{(C-5)} - 3\right). \end{aligned}$$

$f^{-1}(C) =$ $\frac{1}{7}\ln\left(10^{(C-5)} - 3\right)$

b. [2 points] What is the range of $f^{-1}(C)$? Write your final answer *in the space provided*, **using inequalities**.

The range of $f^{-1}(C)$ *is* $g \geq 0$

3. [8 points] **You do not need to show any work for this problem.**

a. [2 points] Which of the following functions dominates **all** the others as $x \to \infty$? **Circle exactly one of the options below.**

[✓ $f(x) = 0.01(1.3)^x$] $g(x) = 100x^{10}$ $h(x) = 300(0.25)^x$

$i(x) = 4^{-2x}$ $j(x) = 300\ln(4|x|)$ $k(x) = 100\left(\frac{6}{5}\right)^x$

b. [2 points] Which of the following functions dominates **all** the others as $x \to -\infty$? **Circle exactly one of the options below.**

$f(x) = 0.01(1.3)^x$ $g(x) = 100x^{10}$ $h(x) = 300(0.25)^x$

[✓ $i(x) = 4^{-2x}$] $j(x) = 300\ln(4|x|)$ $k(x) = 100\left(\frac{6}{5}\right)^x$

c. [2 points] Let $f(x)$ be an odd function with:

$$\lim_{x\to -3^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to -3^-} f(x) = \infty$$

Suppose that $f(3) = 0$. Evaluate $\lim_{x\to 3^-} f(x)$. Write your answer *in the space provided.* If there is not enough information to evaluate the limit, write NOT ENOUGH INFORMATION.

$\lim_{x\to 3^-} f(x) =$ ∞

d. [2 points] Consider the functions:

$$f(x) = 1 + \sqrt{1+x}$$
$$g(x) = 1 + x$$

Find the formula of a function $h(x)$ for which $f(x) = g(h(x))$. Write your answer *in the space provided.*

$h(x) =$ $\sqrt{1+x}$

4. [8 points] In this problem, you should **show your work**. All your answers should be **exact**, and must be found *algebraically*. Write your final answers *in the spaces provided.*

For parts (a) and (b), consider the function

$$F(x) = \frac{(100x^2+3)(x^2+2x-1)}{(x^2-2x-3)(2x^2+4)}$$

a. [2 points] Find the horizontal intercept(s) of $y = F(x)$. If the function has no horizontal intercepts, write NONE in the space provided.

Solution: To find the horizontal intercept of $y = F(x)$, we need to find all values of x for which $(100x^2+3)(x^2+2x-1) = 0$. This means that either $100x^2+3 = 0$ or $x^2+2x-1 = 0$, and the first equation has no solutions since $100x^2+3$ is always at least 3. To solve $x^2+2x-1 = 0$, we use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-2 \pm \sqrt{2^2-4(-1)}}{2} = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}.$$

We're almost done, but we need to make sure that the denominator, $(x^2-2x-3)(2x^2+4)$ isn't zero when $x = -1 \pm \sqrt{2}$. We could just plug these values of x into the denominator and check (with a calculator) that we don't get 0, or notice that the denominator can be factored as $(x-3)(x+1)(2x^2+4)$, and so the denominator can only be zero for $x = 3$ or $x = -1$ (and hence, cannot be zero for $x = -1 \pm \sqrt{2}$).

Horizontal intercept(s): $(-1+\sqrt{2}, 0)$ and $(-1-\sqrt{2}, 0)$

b. [2 points] Find the equation(s) of the horizontal asymptote(s) of $y = F(x)$. If the function has no horizontal asymptotes, write NONE in the space provided.

Solution: The leading term in the numerator is $100x^4$, and the leading term in the denominator is $2x^4$. So:

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{100x^4}{2x^4} = 50$$

Horizontal asymptote(s): $y = 50$

c. [4 points] Consider the function

$$G(x) = \frac{x^2(x^2+5)^3}{(x-2)(x^2+5)^4 x}$$

Find the equation(s) of the vertical asymptote(s) of $y = G(x)$, and the x-coordinate(s) of the hole(s) of $y = G(x)$. If the function has no vertical asymptotes or has no holes, write NONE in the relevant space.

Solution: Note that (x^2+5) is always positive, so we don't need to consider it when finding the holes and the vertical asymptotes. It's then easy to see that $y = G(x)$ has a hole at $x = 0$, and a vertical asymptote at $x = 2$.

Vertical asymptote(s): $x = 2$

x-coordinate(s) of hole(s): 0

5. [12 points] Consider the following expressions:

$9e^{-2x}$	$4x+9$	$9\tan\left(\frac{\pi}{2}x\right)$	$\frac{18}{1+e^{-x}}$
$3x^2+9$	$9+\sin\left(\frac{x}{30}\right)$	$9\tan(3\pi x)$	$\ln(2x-1)$

In each of the following parts, write down *in the space provided* **all** of the expressions above which *could* be formulas for the function described. If more than one expression applies, write them all on the same line and ensure that they are clearly separated. If none of the expressions apply, write NONE OF THE ABOVE.

a. [3 points] The function $f(x)$ satisfies $f(x) \geq 9$ for $x \geq 0$.

$f(x)$ could be: $4x+9$, $\frac{18}{1+e^{-x}}$ or $3x^2+9$

b. [3 points] The function $g(x)$ is periodic with period $\frac{1}{3}$.

$g(x)$ could be: $9\tan(3\pi x)$

c. [3 points] The function $h(x)$ has a vertical asymptote at $x = \frac{1}{2}$.

$h(x)$ could be: $\ln(2x-1)$ or $9\tan(3\pi x)$

d. [3 points] The function $i(x)$ has $\lim_{x\to\infty} i(x) = C$ where $C \geq 0$ is a nonnegative constant.

$i(x)$ could be: $9e^{-2x}$ or $\frac{18}{1+e^{-x}}$

6. [11 points] For this problem, your final answers must be **exact** and should be written *in the spaces provided.*

a. [5 points] Let $V(t)$ be the voltage across a resistor in a circuit (measured in volts) t minutes after 8:00 a.m. on January 29, 2013. The function $V(t)$ is periodic, and it takes 5 minutes to go from a minimum of -10 volts to a maximum of 40 volts. At 8:37 a.m., the voltage across the resistor is -10 volts. Find a formula for $V(t)$, assuming $V(t)$ is a sinusoidal function of t.

Solution: The maximum of $V(t)$ is 40 volts and the minimum -10 volts, so we can calculate the amplitude and the midline:

$$\text{Amplitude} = \frac{40-(-10)}{2} = 25$$

and

$$\text{Midline} : y = \frac{40+(-10)}{2} = 15$$

Since it takes 5 minutes to go from a minimum to a maximum, we know the period is $2 \cdot 5 = 10$. And finally, since the point $(37, -10)$ is a minimum on the graph of $V(t)$, we get:

$$V(t) = -25\cos\left(\frac{2\pi}{10}(t-37)\right) + 15$$

$V(t) =$ $-25\cos\left(\frac{2\pi}{10}(t-37)\right) + 15$

b. [6 points] Find all values of t in the interval $-0.5 \le t \le 1$ for which:

$$5\sin\left(2\pi\left(t+\frac{1}{4}\right)\right) + 3 = 0$$

Your answer must be found *algebraically* and should be **exact**. You must **show your work** carefully to receive full credit.

Solution: We first isolate the sine to get:

$$\begin{aligned} 5\sin\left(2\pi\left(t+\frac{1}{4}\right)\right) + 3 &= 0 \\ \sin\left(2\pi\left(t+\frac{1}{4}\right)\right) &= -0.6 \end{aligned}$$

Two 'different' solutions to $\sin(x) = -0.6$ are given by $x = \sin^{-1}(-0.6)$ and $x = \pi - \sin^{-1}(-0.6)$, and so we can get two 'different' solutions to the equation above by setting:

$$\begin{aligned} 2\pi(t+0.25) &= \sin^{-1}(-0.6) \\ t+0.25 &= \frac{\sin^{-1}(-0.6)}{2\pi} \\ t &= \frac{\sin^{-1}(-0.6)}{2\pi} - 0.25 \end{aligned}$$

$$\begin{aligned} 2\pi(t+0.25) &= \pi - \sin^{-1}(-0.6) \\ t+0.25 &= 0.5 - \frac{\sin^{-1}(-0.6)}{2\pi} \\ t &= 0.25 - \frac{\sin^{-1}(-0.6)}{2\pi} \end{aligned}$$

Finally, we need to add/subtract the period to get the solutions in the interval $[-0.5, 1]$. Doing this with a calculator gives us our final answer (below).

The solutions in $-0.5 \le t \le 1$ *are* $\frac{\sin^{-1}(-0.6)}{2\pi} - 0.25; -\frac{\sin^{-1}(-0.6)}{2\pi} + 0.25; \frac{\sin^{-1}(-0.6)}{2\pi} + 0.75$

7. [8 points] Some values of the function $f(x)$ are given in the table below.

x	-2	0	2	5	8	10	15	18	20	21
$f(x)$	54.22	30.50	17.16	7.24	13.84	18.24	29.24	5	-9	-20

Note that all values in the table have been rounded to two decimal places. You must **show your work** for each part of this problem, and write your final answers *in the spaces provided.*

a. [3 points] Find a formula for $f(x)$ valid for $-2 \le x \le 5$, assuming that f is exponential on the interval $[-2, 5]$.

Solution: Since $f(x)$ is exponential, we know that it has the form $f(x) = ab^x$. From the table above, we see immediately that $a = f(0) = 30.50$. To get b, we use the fact that $f(2) = 17.16$ to get:

$$\begin{aligned} 30.5b^2 &= 17.16 \\ b^2 &= \frac{17.16}{30.5} \\ b &= \sqrt{\frac{17.16}{30.5}} \end{aligned}$$

$f(x) =$ $30.5\left(\sqrt{\frac{17.16}{30.5}}\right)^x$

b. [2 points] Find a formula for $f(x)$ valid for $5 \le x \le 15$, assuming that $f(x)$ is linear on the interval $[5, 15]$.

Solution: Since $f(x)$ is linear, it has the form $f(x) = mx + b$. We find the slope using the points $(5, 7.24)$ and $(8, 13.84)$:

$$m = \frac{13.84 - 7.24}{8 - 5} = \frac{6.6}{3} = 2.2.$$

So we get $f(x) - 7.24 = 2.2(x - 5)$, and hence $f(x) = 2.2(x - 5) + 7.24$.

$f(x) =$ $2.2(x - 5) + 7.24$

c. [3 points] Show that $f(x)$ cannot be concave down on the interval $[15, 21]$. Make sure any relevant calculations are clearly shown, and write a *brief* sentence explaining your reasoning.

Solution: Calculating the average rate of change of $f(x)$ on the intervals $[15, 18]$ and $[18, 20]$ gives us:

$$\frac{5 - 29.24}{18 - 15} = \frac{-24.24}{3} = -8.08$$

and

$$\frac{-9 - 5}{20 - 18} = \frac{-14}{2} = -7.$$

Since the average rate of change on $[18, 20]$ is greater than the average rate of change on $[15, 18]$, $f(x)$ cannot be concave down on $[15, 21]$.

8. [15 points] The number of hemlock trees in the southern Appalachian mountains is declining as a result of an infestation of hemlock woolly adelgids (a kind of insect).

- There are $H(d)$ *healthy* hemlock trees in the southern Appalachian mountains d days after January 1, 2013.
- There are $I(d)$ *infested* hemlock trees in the southern Appalachian mountains d days after January 1, 2013.

Note that all hemlock trees are considered healthy unless they are infested. Be sure to write your final answers *in the spaces provided*.

a. [2 points] Let $J(w)$ be the number of *healthy* hemlock trees in the southern Appalachian mountains *w weeks* after January 1, 2013. Find a formula for $J(w)$ in terms of the functions H or I (or possibly both).

$J(w) =$ $H(7w)$

b. [3 points] Let $F(d)$ be the fraction of the hemlock trees in the southern Appalachian mountains that are *infested* d days after January 1, 2013. Find a formula for $F(d)$ in terms of the functions H or I (or possibly both).

$F(d) =$ $\frac{I(d)}{H(d)+I(d)}$

c. [4 points] Let $K(d)$ be the total number of hemlock trees in the southern Appalachian mountains, in *thousands*, d days after January 1, 2013. Find a formula for $K(d)$ in terms of the functions H or I (or possibly both).

$K(d) =$ $0.001(H(d)+I(d))$

THIS PROBLEM CONTINUES ON THE NEXT PAGE

d. [3 points] The number of hemlock trees I that are *infested* in the southern Appalachian mountains is *inversely proportional* to the cube of the total amount of money M (in millions of dollars) that the government spends combating the spread of the adelgids. Write a formula for I in terms of M, assuming that there were 2,000 infested trees when the government had spent 3 million dollars. You must **show your work** for this part.

Solution: Since I is inversely proportional to the cube of M, we have:

$$I = \frac{k}{M^3}$$

We know that $I = 2000$ when $M = 3$, which gives us:

$$2{,}000 = \frac{k}{3^3}$$

and so we have $k = 3^3 \cdot 2{,}000 = 162{,}000$.

$I =$ $\frac{162,000}{M^3}$

e. [3 points] The number of hemlock woolly adelgids $A(M)$ (in millions) is also a function of the amount of money M (in millions of dollars) that the government spends to try to preserve the hemlock trees, and is given by:

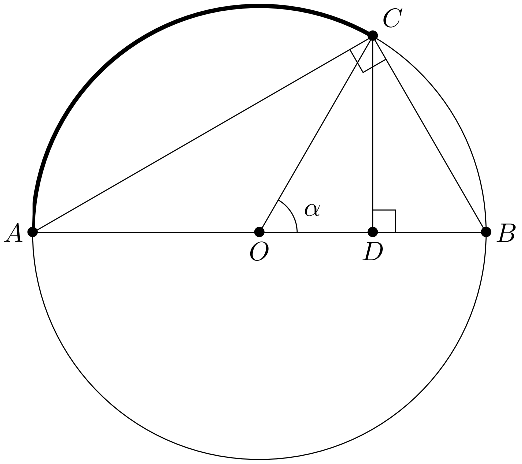
$$A(M) = \frac{4}{M}$$

for $M \geq 4$. Find the equation of the horizontal asymptote of $y = A(M)$, **and** interpret this horizontal asymptote in practical terms.

Solution: It's easy to see that the horizontal asymptote is $y = 0$. Practically, this means that as the amount of money the government spends increases (going to ∞), the number of hemlock woolly adelgids in the southern Appalachian mountains approaches 0.

The equation of the horizontal asymptote is $y = 0$

9. [7 points] Consider the circle of radius R centered at the point O, illustrated below. Note that the diagram is not drawn to scale.



Note that the line AB contains the point O, and the angles ACB and ADC both have measure $\frac{\pi}{2}$ radians. α is the positive measure of the angle COD (see the dagram), while L is the length of the line segment AC.

a. [2 points] Find the length of the line segment CD. Your answer for this part may involve any or all of the constants R, L and α.

The length of CD *is* $R\sin\alpha$

b. [3 points] Find the (positive) measure of the angle OAC in radians. Your answer for this part may involve the constants R and L, but must **not** include the constant α.

Solution: Let β be the measure of the angle OAC in radians. Note that AB has length $2R$, so in the right triangle ACB, we have:

$$\cos\beta = \frac{L}{2R}$$

and hence

$$\beta = \cos^{-1}\left(\frac{L}{2R}\right)$$

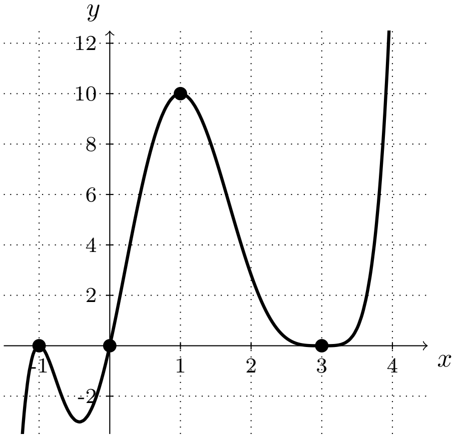
The measure of OAC *is* $\cos^{-1}\left(\frac{L}{2R}\right)$

c. [2 points] Find the length of the (bolded) circular arc AC. Your answer for this part may involve any or all of the constants R, L and α.

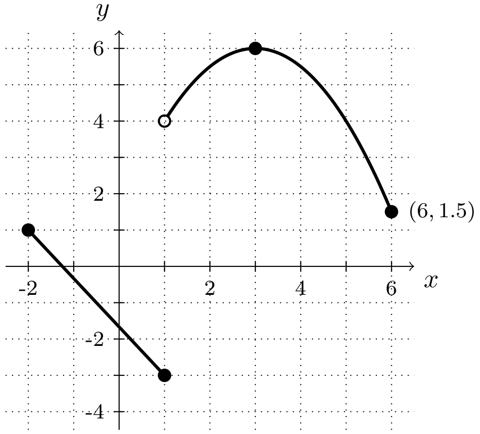
The length of the arc AC *is* $R(\pi-\alpha)$

10. [11 points] Consider the graphs of $y = k(x)$ and $y = \ell(x)$ given below:

GRAPH OF $y = k(x)$



GRAPH OF $y = \ell(x)$



You must **show your work** in both parts of this problem to receive full credit. Write your final answers *in the spaces provided.*

a. [5 points] Find a formula for $k(x)$, assuming $k(x)$ is a polynomial of degree seven with zeros at $x = -1$, $x = 0$ and $x = 3$.

Solution: The zeros of $k(x)$ are $x = -1$, 0 and 3. From the graph, we can see that the zero at $x = 0$ has multiplicity one, and the zeroes at $x = -1$ and $x = 3$ have even multiplicities. The graph is clearly flatter at $x = 3$, so the multiplicity at this zero is larger than the multiplicity of the zero at $x = -1$. Since we know $k(x)$ has degree 7, we know that the multiplicity of the zero at $x = -1$ must be 2, and the multiplicity at $x = 3$ must be 4, giving us $k(x) = ax(x + 1)^2(x - 3)^4$ for some constant a.

Since the point $(1, 10)$ is on the graph, we know that we must have $k(1) = 10$, and hence $10 = a(2)^2(-2)^4$, giving us $a = \frac{5}{32}$.

$$k(x) = \underline{\frac{5}{32}x(x+1)^2(x-3)^4}$$

b. [6 points] Find a piecewise-defined formula for $\ell(x)$ on $[-2, 6]$, given that the graph of $y = \ell(x)$ is made up of a line and a parabola.

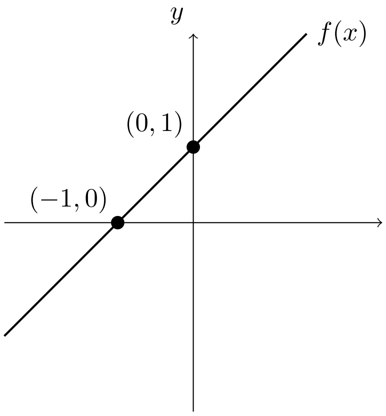
Solution: We see from the graph that the linear part has slope

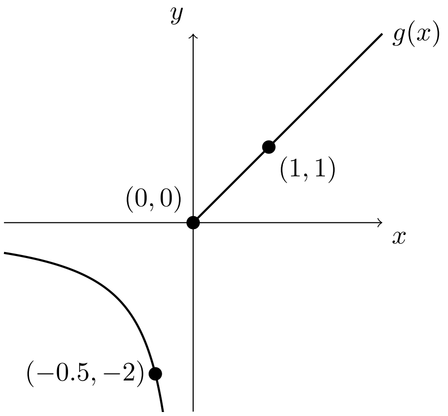
$$\frac{-3-1}{1-(-2)} = -\frac{4}{3}$$

So for the linear part, we must have $\ell(x) - (-3) = -\frac{4}{3}(x - 1)$, and hence $\ell(x) = -\frac{4}{3}(x - 1) - 3$. The quadratic part clearly has a vertex at $(3, 6)$, and so we must have $\ell(x) = a(x - 3)^2 + 6$ for some constant a. Since the point $(6, 1.5)$ is on the quadratic part, we get $1.5 = a(3)^2 + 6$, and so $a = -0.5$.

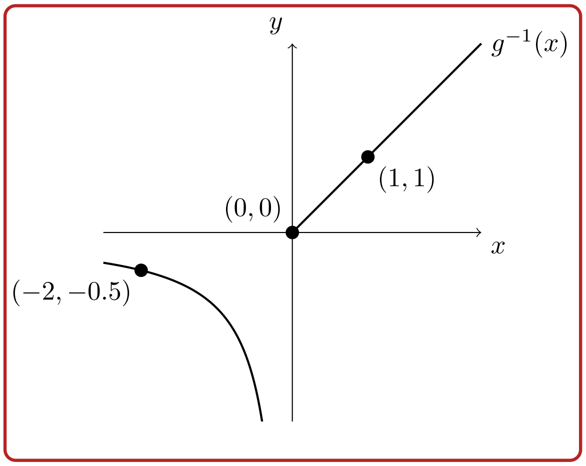
$$\ell(x) = \begin{cases} \underline{-\frac{4}{3}(x-1) - 3} & \text{if } \underline{-2 \le x \le 1} \\ \underline{-0.5(x-3)^2 + 6} & \text{if } \underline{1 < x \le 6} \end{cases}$$

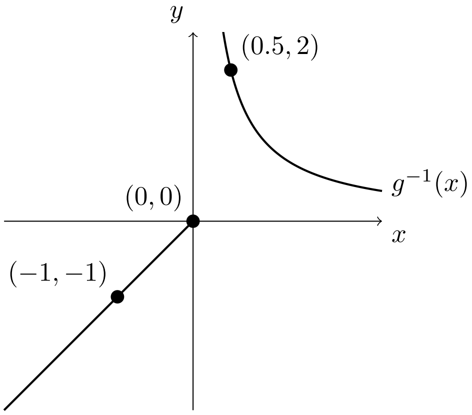
11. [5 points] A portion of the graphs of $y = f(x)$ and $y = g(x)$ are given below. **You do not need to show any work for this problem.**

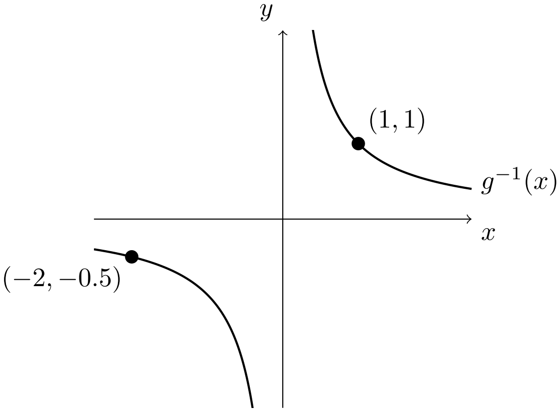


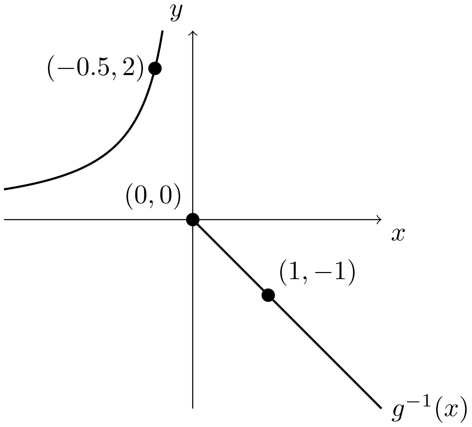


a. [2 points] Assume that $g(x)$ is an invertible function. Which of the following could be the graph of $y = g^{-1}(x)$? **Circle exactly one of the four graphs below**.









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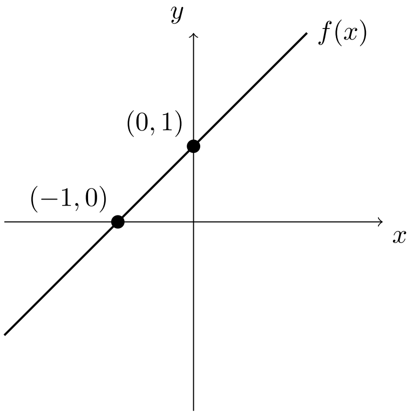
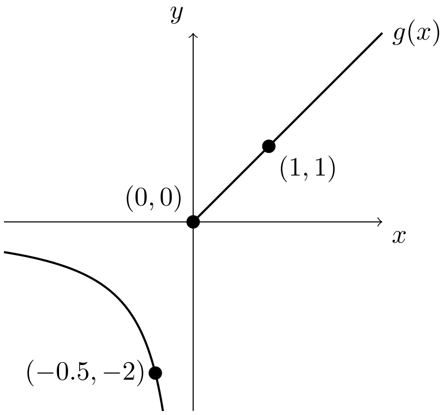
The graphs of $y = f(x)$ *and* $y = g(x)$ *from the previous page have been reproduced below for your convenience.*





b. [3 points] Which of the following could be the graph of $y = g(f(x))$? **Circle exactly one of the four graphs below**.