Math 105 — First Midterm

October 10, 2017

UMID: EXAM SOLUTIONS Initials: ______

Instructor: ______ Section: ______

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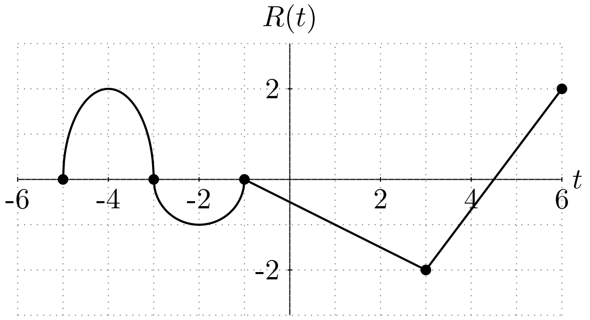
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	14	
2	11	
3	14	
4	12	
5	13	
6	9	
7	11	
8	6	
9	10	
Total	100	

1. [14 points] The graph of a function $R(t)$ is given below.



You do not need to show your work on this problem.

a. [3 points] Using **interval** notation, write the t-values where R is increasing and the t-values where R is decreasing.

R is increasing on $[-5,-4] \cup [-2,-1] \cup [3,6]$

R is decreasing on $[-4,-2] \cup [-1,3]$

b. [3 points] Using **inequalities**, write the t-values where R is concave up and the t-values where R is concave down.

R is concave up for $-3 \leq t \leq -1$

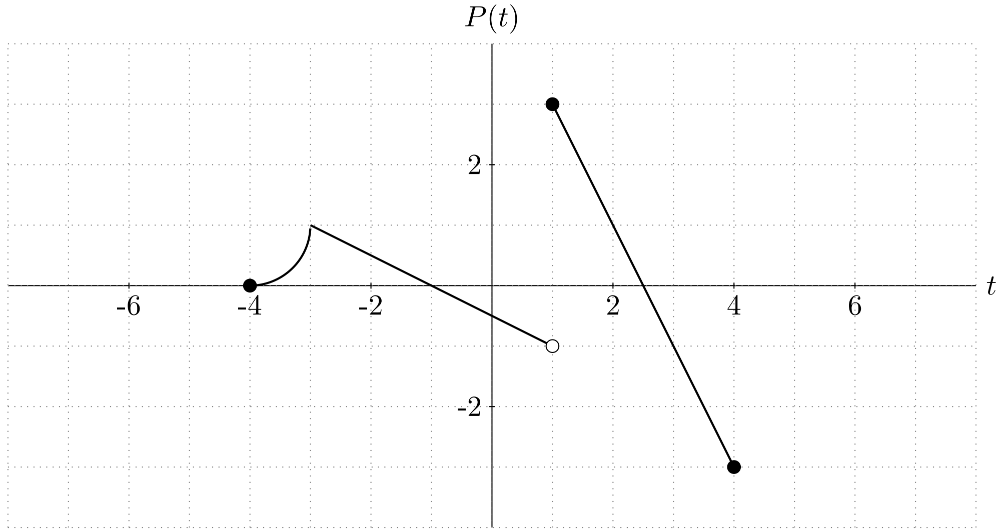
R is concave down for $-5 \leq t \leq -3$

c. [2 points] Find the average rate of change of R between $t = -2$ and $t = 3$.

The average rate of change is $\dfrac{-2-(-1)}{3-(-2)} = -\dfrac{1}{5}$

d. [6 points] On the axes below, sketch a well-labeled graph of the piecewise-defined function

$$P(t) = \begin{cases} R(t+2)+1 & \text{for } -4 \leq t < 1 \\ -2t+5 & \text{for } 1 \leq t \leq 4. \end{cases}$$



2. [11 points] Gretchken's experiments have quickly depleted Chuck's egg supply, and he has to buy more eggs from the wholesaler. Chuck has \$1050 to spend, and the wholesaler informs him that the price of mealworm eggs and cricket eggs are now \$6.50 and \$8 per pound, respectively.

a. [4 points] Let f be the function that gives the amount M of mealworm eggs in pounds that Chuck can afford if he buys C pounds of cricket eggs (in other words, we have $M = f(C)$). Write a formula for the function f.

Solution: We have $6.5M + 8C = 1050$. Rearranging gives $M = \frac{2}{13}(1050 - 8C)$.

$f(C) =$ $\frac{2}{13}(1050 - 8C)$

After some bargaining, the wholesaler gives Chuck a **special offer**. If he buys \$400 worth of cricket eggs at \$8 per pound, then he will be charged only \$7.50 for each subsequent pound of cricket eggs beyond the first \$400.

b. [2 points] If Chuck spends \$400 on cricket eggs, what amount of mealworm eggs can he buy? Circle your final answer.

Solution: If Chuck spends \$400 on cricket eggs, he buys 50 pounds of them. We compute $f(50) = 2/13 \cdot (1050 - 400) = 100$. He can buy 100 pounds of mealworm eggs.

c. [5 points] Let g be the new function that gives the amount M of mealworm eggs in pounds that Chuck can afford if he buys C pounds of cricket eggs with the **special offer**. Write a piecewise-defined formula for the function g.

Solution: There is no change in price if Chuck buys less than 50 pounds of cricket eggs, so $g(C) = f(C)$ for $0 \leq C \leq 50$. Thereafter, the function is linear with slope $-7.5/6.5 = -15/13$. By the previous part, we know that the graph passes through the point $(50, 100)$. The formula on this stretch is $g(C) = -15/13 \cdot (C - 50) + 100$. Finally, we need to find the right endpoint for the domain of g:

$$\frac{-15}{13}(C - 50) + 100 = 0$$

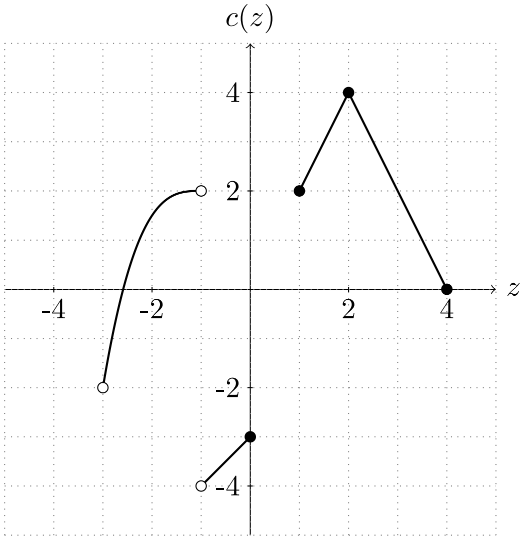
$$C = \frac{1300}{15} + 50 = \frac{410}{3}$$

$$g(C) = \begin{cases} f(C) & \text{for } 0 \leq C \leq 50 \\ -15/13 \cdot (C - 50) + 100 & \text{for } 50 \leq C \leq 410/3 \end{cases}$$

3. [14 points] Consider the functions $a(y)$, $b(w)$ and $c(z)$ given below.

y	-10	-4	-1	1	3	4
$a(y)$	4	-2	2	-4	0	3

$$b(w) = \begin{cases} 1.5w + 8 & \text{for } -5 \le w < -1 \\ -4 \cdot 2^{-w} & \text{for } 1 \le w \le 5. \end{cases}$$



a. [3 points] Find the domain of $c(z)$. Express your answer in interval notation or using inequalities.

The domain of $c(z)$ is $(-3,-1) \cup (-1,0] \cup [1,4]$

b. [3 points] Find the range of $b(w)$. Express your answer in interval notation or using inequalities.

The range of $b(w)$ is $[-2,-1/8] \cup [1/2,13/2)$

c. [4 points] Calculate the following or write "UNDEFINED" if the quantity is not defined. Simplify your answer.

(i) $(a(-1))^{-1} =$ $1/2$

(ii) $a(a(-10)) =$ 3

(iii) $c(b(-5) + 2.5) =$ 2

(iv) $b^{-1}(2) =$ -4

d. [4 points] Using only the information given, find all solutions to each of the equations below. Simplify your answers, but leave them in **exact** form. If an equation has no solution, write "NO SOLUTION" in the blank.

(i) $c(a(y)) = 2$.

$y =$ 4

(ii) $b(w) = a(3)$.

$w =$ NO SOLUTION

4. [12 points] Chump is on his yacht, enjoying his annual vacation. After finishing a bottle of Martinelli's sparkling apple cider, he tosses the empty bottle into the ocean. The trajectory of the bottle is a parabola. When the bottle is a horizontal distance of x meters away from Chump, it is $H(x)$ meters above the level of the yacht deck, where $H(x) = -x^2 + \frac{\pi}{2}x + \frac{1}{2}$.

a. [5 points] Use the method of completing the square to put $H(x)$ in vertex form. **Your answer must be exact**, and you must *show all your work, step-by-step*, to get full credit.

Solution:

$$\begin{aligned} H(x) &= -x^2 + \frac{\pi}{2}x + \frac{1}{2} \\ &= -\left(x^2 - \frac{\pi}{2}x\right) + \frac{1}{2} \\ &= -\left(x^2 - \frac{\pi}{2}x + \frac{\pi^2}{16} - \frac{\pi^2}{16}\right) + \frac{1}{2} \\ &= -\left(\left(x - \frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right) + \frac{1}{2} \\ &= -\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right) \end{aligned}$$

$H(x) =$ $-\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right)$

b. [2 points] What was the maximum height of the bottle? Give your answer in exact form.

The maximum height was $\frac{\pi^2}{16} + \frac{1}{2}$ meters above the level of the yacht deck.

c. [5 points] Suppose the deck of the yacht is 1 meter above the surface of the ocean. What is the horizontal distance between Chump and the bottle when it hits the ocean? Leave your answer in exact form.

Solution: When the bottle hits the ocean, we have $H(x) = -1$, or

$$-\left(x - \frac{\pi}{4}\right)^2 + \left(\frac{\pi^2}{16} + \frac{1}{2}\right) = -1$$

We solve this equation for x as follows.

$$\left(x - \frac{\pi}{4}\right)^2 = \frac{\pi^2}{16} + \frac{1}{2} + 1 = \frac{\pi^2}{16} + \frac{3}{2}$$

$$x = \frac{\pi}{4} + \sqrt{\frac{\pi^2}{16} + \frac{3}{2}} \quad \text{or} \quad x = \frac{\pi}{4} - \sqrt{\frac{\pi^2}{16} + \frac{3}{2}}.$$

Only the first solution is positive, so this is the one we want.

The bottle was a horizontal distance of $\underline{\dfrac{\pi}{4} + \sqrt{\dfrac{\pi^2}{16} + \dfrac{3}{2}}}$ meters away from Chump.

5. [13 points] After seeing the good effects of Gretchken's running routine, Chuck has decided to start running as well. Suppose $C(d)$ is the time (in seconds) it takes Chuck to run d meters, and suppose $G(d)$ is the time (in seconds) it takes for Gretchken to run d meters. Suppose C and G both have inverse functions.

a. [3 points] Give a practical interpretation of the expression $G^{-1}(600) = 800$.

Solution:
Gretchken takes 600 seconds to run 800 meters.

b. [4 points] Give a practical interpretation of the expression $C^{-1}(G(300)) = 200$.

Solution: It takes Chuck the same amount of time to run 200 meters as it takes Gretchken to run 300 meters.

c. [3 points] Give an expression using function notation for Chuck's average speed in meters per second during his first 720 seconds of running. Circle your final answer.

Solution:

$$\frac{C^{-1}(720)}{720}$$

d. [3 points] If $D(h)$ is the distance in meters Chuck needs to run to burn h calories, give a practical interpretation of the quantity $C(D(100))$.

Solution: $C(D(100))$ is the amount of time in seconds it takes Chuck to burn 100 calories while running.

6. [9 points] Chuck made a new mathematical model to relate the time of the day and the number of customers at the farmers' market. His model $N(t)$, predicts the number of customers at the market t hours after 6:30 am, the time at which he normally arrives. We have the following table of values for $N(t)$.

t	0	3	4	5	7
$N(t)$	132	105	120	124	88

Between consecutive values of t in the table, assume that $N(t)$ is either only increasing or only decreasing, and assume that it does not change concavity between consecutive t-values in the table. Also assume that the domain of $N(t)$ is $[0, 7]$.

a. [2 points] What is the largest interval over which $N(t)$ could be concave up? Circle your final answer.

Solution:

$$[0, 4]$$

b. [2 points] What is the largest interval over which $N(t)$ could be concave down? Circle your final answer.

Solution:

$$[3, 7]$$

c. [5 points] On one particular Saturday, Chuck learns that there will be a group of 25 additional customers arriving at the market at 10:45 am and leaving at 12:30 pm. He wishes to write a function $P(t)$ to model the number of customers at the market t hours after his arrival on this particular Saturday. Write a piecewise-defined formula for $P(t)$ in terms of the original model $N(t)$. Circle your final answer.

Solution: At 10:45 am, we have $t = 4.25$ and at 12:30 pm, we have $t = 6$. We therefore have the formula

$$P(t) = \begin{cases} N(t) & 0 \le t < 4.25 \\ N(t) + 25 & 4.25 \le t < 6 \\ N(t) & 6 \le t \le 7. \end{cases}$$

7. [11 points] Gretchken has managed to synthesize an even more powerful growth stimulant, Chemical Y. She administers it to a freshly hatched mealworm, and observes the mealworm's growth over the next few days. Let $M(t)$ denote the mass (in grams) of the mealworm t weeks after it hatches. Gretchken makes the following measurements. You do not have to show your work for this problem.

t	0	3	5
$M(t)$	18	24	32

a. [3 points] What type of function COULD $M(t)$ be? Circle *all* that apply. If none apply, circle "none of these".

linear [✓ quadratic] exponential none of these

b. [4 points] Gretchken next tests Chemical Y on a silkworm. Let $S(t)$ be the mass (in grams) of the silkworm t weeks after it hatches. Give a practical interpretation of $S(t) = M(t+2)$.

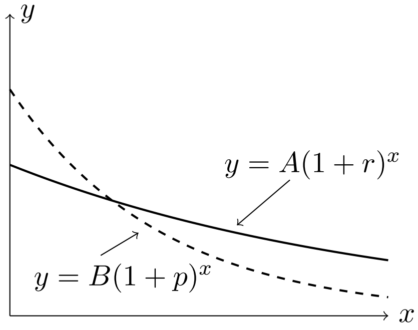
Solution:
When both are on Chemical Y, the mass of a silkworm is equal to the mass of a mealworm that has hatched 2 weeks earlier.

c. [4 points] Gretchken tests Chemical Y on a cockroach. The cockroach weighs $C(t)$ grams t weeks after it hatches. Gretchken has found that $C(t)$ has the formula $C(t) = 2(1.3)^{2.5t-2}$. Leave your answers in exact form.

(i) The weekly growth factor of $C(t)$ is $\underline{\quad 1.3^{2.5} \quad}$.

(ii) The vertical intercept of $C(t)$ is $\underline{\quad 2 \cdot 1.3^{-2} \quad}$.

8. [6 points] Consider the following graph of two functions with their formulas given. The letters A, B, r, p are all constants.



Compare the two quantities given by putting one of the symbols ">", "<", or "=" in the blank provided. If the relationship between the quantities cannot be determined, write "N" in the blank. You do not have to show your work.

(i) $A \;\underline{\quad < \quad}\; B$

(ii) $r \;\underline{\quad > \quad}\; p$

(iii) $\lim_{x\to\infty} A(1+r)^x \;\underline{\quad = \quad}\; \lim_{x\to\infty} B(1+p)^x$

9. [10 points] Suppose $L(t)$ is a linear function, $Q(t)$ is a quadratic function, and $E(t)$ is an exponential function, each with domain all real numbers. Also, assume that $E(3) = 1$. For each of the following statements, circle the correct option.

a. [2 points] The graphs of $E(t)$ and $L(t)$ intersect exactly once.

must be true [✓ could be true] never true

b. [2 points] $E(-1)$ is negative.

must be true could be true [✓ never true]

c. [2 points] The graphs of $Q(t)$ and $L(t)$ intersect exactly twice.

must be true [✓ could be true] never true

d. [2 points] The graph of $E(t)$ is concave up.

[✓ must be true] could be true never true

e. [2 points] The graphs of $Q(t)$ and $Q(t+2) - 5$ intersect exactly twice.

must be true could be true [✓ never true]