Math 105 — Final Exam

December 14, 2017

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	8	
2	16	
3	10	
4	11	
5	9	
6	11	
7	15	
8	5	
9	5	
10	10	
Total	100	

1. [8 points] For both parts of this question, please leave your answers in **exact form** and show **all** your work.

a. [3 points] Find all values of w satisfying the following equation.

$$\ln(10w + 3) = 8.$$

Solution: Take the base e exponential of both sides to get

$$10w + 3 = e^8.$$

We can then solve this for w as follows:

$$10w = e^8 - 3$$

$$w = \frac{e^8 - 3}{10}$$

$w =$ $\dfrac{e^8 - 3}{10}$.

b. [5 points] Find all values of x with $-2 \le x \le 1$ satisfying the following equation.

$$4\tan\left(\frac{2\pi}{3}x\right) = 5.$$

Solution: To find the principal solution, we solve

$$\tan\left(\frac{2\pi}{3}x\right) = 5/4,$$

$$\frac{2\pi}{3}x = \tan^{-1}(5/4).$$

Hence,

$$x = \frac{3}{2\pi}\tan^{-1}(5/4) \approx 0.428.$$

To find the other solutions, observe that the period of the function is $\pi \cdot \frac{3}{2\pi} = 3/2$. Hence, we also have a solution at

$$x = \frac{3}{2\pi}\tan^{-1}(5/4) - \frac{3}{2}.$$

$x =$ $\dfrac{3}{2\pi}\tan^{-1}(5/4), \quad \dfrac{3}{2\pi}\tan^{-1}(5/4) - \dfrac{3}{2}$.

2. [16 points] The five parts of this question are **unrelated** to each other.

a. [3 points] $f(x)$ is a periodic function with domain $(-\infty, \infty)$, with a period of 5, with an amplitude of 2, and with midline $y = -1$. Find the amplitude, period and midline of $g(x) = -7f(2(x-3)) + 1$.

The period of $g(x)$ is ______ 5/2 ______.

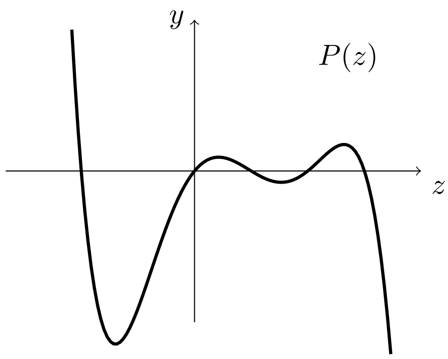
The amplitude of $g(x)$ is ______ 14 ______.

The midline of $g(x)$ is ______ $y = 8$ ______.

b. [2 points] Let $a, b, c > 0$ be positive constants. Evaluate the following limit. You do not need to show any work for this part.

$$\lim_{x\to\infty} \frac{(\sqrt{x}+2)^6}{ax^3+bx+c} = \underline{\quad 1/a \quad}.$$

c. [3 points] The graph of a polynomial $y = P(z)$ **with its end behavior shown** is graphed below. Answer the following questions about $P(z)$.



Is the degree of $P(z)$ even or odd?

even [odd] not possible to tell

Is the leading coefficient of $P(z)$ positive or negative?

positive [negative] not possible to tell

What is the smallest possible degree of $P(z)$?

The smallest possible degree of $P(z)$ is ______ 5 ______.

2. (continued) Reminder: The parts of this question are **unrelated** to each other.

d. [3 points] The table below give some values of a function $M(t)$ at different t-values.

t	0	3	5
$M(t)$	625	900	1296

What type of function could $M(t)$ be? Circle all that apply.

linear exponential [quadratic] none of these

Could $M(t)$ be proportional to t^2?

yes [no]

If you answered no, briefly explain why not, and if you answered yes, find the constant of proportionality.

Solution: To be proportional to a power function, we would need $M(0) = 0$.

e. [5 points] Consider the function, $H(t) = 3e^{1.2t-3}$, that gives the weight, in grams of a mealworm t days after it hatches. Find the weight of a mealworm when it hatches, find the daily (non-continuous) growth rate of $H(t)$, and find the amount of time it takes for a mealworm to triple in weight, all in **exact form**, with **appropriate units**. Be sure to show **all** your work.

Solution: We want to find t such that $H(t) = 3H(0)$, i.e.

$$3e^{1.2t-3} = 3 \cdot 3e^{-3}.$$

Divide both sides by $3e^{-3}$ to get

$$e^{1.2t} = 3.$$

Take the logarithm both sides, and solve to get

$$1.2t = \ln 3$$

$$t = \frac{\ln 3}{1.2}.$$

The weight when it hatches is $3e^{-3}$ grams.

The daily (not continuous) growth rate is $e^{1.2} - 1$.

The time it takes to triple in weight is $\frac{\ln 3}{1.2}$ days.

3. [10 points] Desperate, Chump enlisted Chuck and Samsa's help in searching for Mrs Chump. The two of them flew high and low, and finally tracked Gregor and Mrs Chump to a cave in the side of Mt. Eggerest. Neither Chuck nor Samsa were strong enough to subdue Gregor, so they planned to rescue Mrs Chump when Gregor fell asleep.

a. [5 points] Gregor's sleepiness level (measured in *snores*) follows a 24 hour cycle. His minimum sleepiness is 2.7 snores at 9:30 pm every day, and his maximum sleepiness is 8.9 snores at 9:30 am. Let $W(t)$ be a sinusoidal function modeling Gregor's sleepiness level (in snores) t **hours after 5 pm on Friday**. Find a formula for $W(t)$.

Solution: The midline of $W(t)$ is $y = (8.9+2.7)/2 = 5.8$, its amplitude is $(8.9-2.7)/2 = 3.1$, and its period is 24. As such, we see that

$$W(t) = 3.1\cos(\frac{2\pi}{24}(t-h)) + 5.8$$

where h is some shift. To compute h, notice that 9:30 am is 16.5 hours after 5 pm, and so $W(t)$ is at a maximum at $t = 16.5$. Hence, we need to shift the graph 16.5 units to the right, and $h = 16.5$.

$W(t) =$ $3.1\cos(\frac{\pi}{12}(t-16.5)) + 5.8$.

b. [5 points] At 7 am, Chuck made his move and slipped quietly into Gregor's cave. To his horror, Gregor sensed movement and stirred awake, leaving Chuck with no choice but to inject him with Gretchken's cockroach tranquilizer. Due to Gregor's strong constitution, he did not fall immediately asleep. Instead, his sleepiness level (in snores) t **minutes** after being injected was

$$I(t) = -2\sin\left(\frac{\pi}{8}(t+2)\right) + 7.$$

How long did Chuck have to wait after the injection before Gregor's sleepiness level rose above 8 snores **for the first time**? Leave your answer in **exact** form, and include **appropriate units**.

Solution: We first solve for the principal value.

$$-2\sin\left(\frac{\pi}{8}(t+2)\right) + 7 = 8$$

$$\sin\left(\frac{\pi}{8}(t+2)\right) = -\frac{1}{2}$$

$$\frac{\pi}{8}(t+2) = \sin^{-1}(-1/2)$$

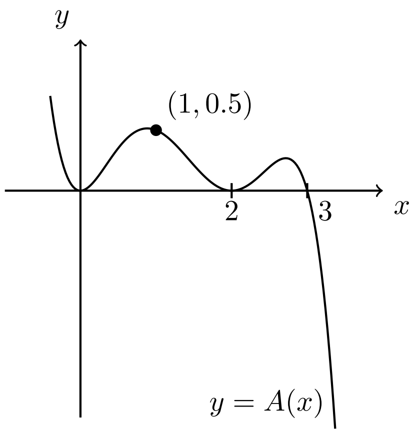
$$t = \frac{8}{\pi}\sin^{-1}(-1/2) - 2 = -\frac{10}{3}.$$

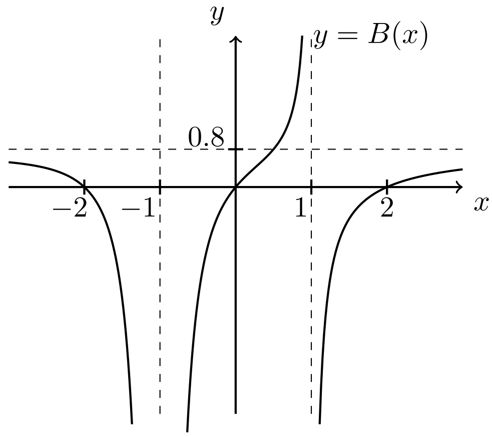
This however, is not the solution we want. We instead want to find the smallest *positive* solution. By inspecting the graph of the function, the solution we want is

$$-2 + 8 - \frac{8}{\pi}\sin^{-1}(-1/2) = 6 + \frac{4}{3} = \frac{22}{3}.$$

Chuck had to wait $\frac{22}{3}$ minutes.

4. [11 points] Consider the graphs of $y = A(x)$ and $y = B(x)$ given below:





a. [2 points] $A(x)$ is a degree 5 polynomial. Write down all of its zeros.

$A(x)$ has zeros at $x =$ $0,\ 2,\ 3$.

b. [3 points] Write down a formula for $A(x)$, showing **all** your work.

Solution: We see that $A(x)$ has double roots at $x = 0, 2$ and a single root at $x = 3$. It thus has the formula

$$A(x) = ax^2(x-2)^2(x-3)$$

To solve for a, we plug in $x = 1$, to get

$$0.5 = a(1)^2(-1)^2(-2),$$

so $a = -\frac{1}{4}$.

$A(x) =$ $-\dfrac{1}{4}x^2(x-2)^2(x-3)$.

c. [3 points] The graph of $B(x)$ has vertical asymptotes at $x = -1$ and $x = 1$, and a horizontal asymptote at $y = 0.8$. If $B(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials, write down all the zeros of both polynomials.

$p(x)$ has zeros at $x =$ $-2,\ 0,\ 2$.

$q(x)$ has zeros at $x =$ $-1,\ 1$.

d. [3 points] Write down a possible formula for $B(x)$.

$B(x) =$ $\dfrac{0.8(x+2)x(x-2)}{(x+1)^2(x-1)}$.

5. [9 points] Chuck and Samsa rescued Mrs Chump and brought her back safely to Chickenville. "I'm sorry," said Chump, apologizing for the first time in his life. Seeing that the cockroaches were not the poultry-eating monsters he thought they were, Chump released all the cockroaches that had been caught. To make further amends, Chump decided to purchase dried leaves from out of town for the cockroaches to eat. t days after February 1, Chump was helping to feed N cockroaches. Some values for t and N are given in the following table.

t	4	6
N	100	1000

a. [4 points] Suppose $N = g(t)$ where $g(t)$ is an exponential function. Find a formula for $g(t)$, leaving your answer in **exact form** and showing **all** your work.

Solution: We have the general formula $g(t) = ab^t$. We solve for a and b as follows. Plugging in $t = 4$ gives

$$100 = g(4) = ab^4.$$

Plugging in $t = 6$ gives

$$1000 = g(6) = ab^6.$$

Dividing these two equations gives $b^2 = 10 \implies b = \sqrt{10}$. Now use this value of b to solve for a. We have

$$100 = a(\sqrt{10})^4 = a100$$

so $a = 1$.

$g(t) =$ $10^{t/2}$.

b. [5 points] Suppose $N = h(t)$ where $h(t)$ is a power function. Find a formula for $h(t)$, leaving your answer in **exact form** and showing **all** your work.

Solution: We have the general formula $g(t) = at^r$. We solve for a and r as follows. Plugging in $t = 4$ gives

$$100 = g(4) = a4^r$$

Plugging in $t = 6$ gives

$$1000 = g(6) = a6^r$$

Dividing these two equations gives $(6/4)^r = 10$. We take logarithms of both sides, and solve to get

$$r \log(3/2) = \log 10 = 1$$

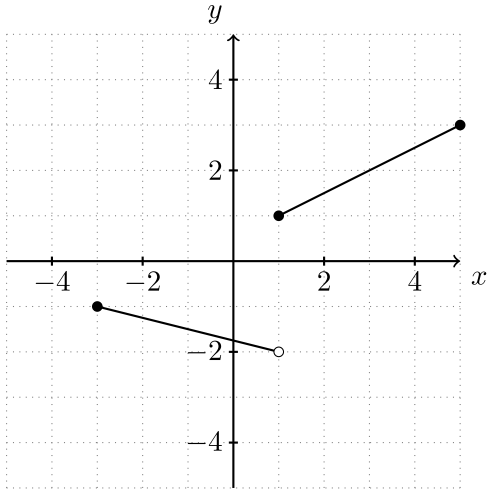
so $r = 1/\log(3/2)$. Now use this value of r to solve for a. We have

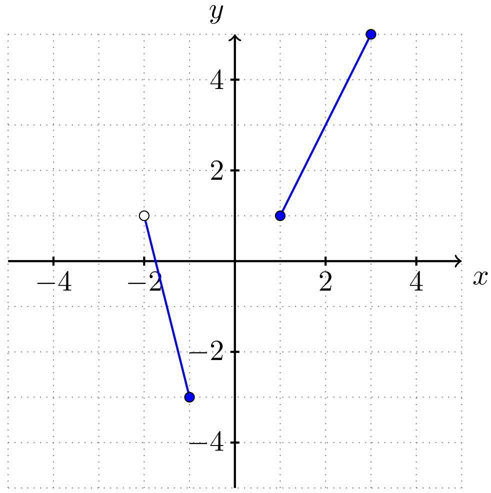
$$100 = a4^{1/\log(3/2)}$$

$$a = 100 \cdot 4^{-1/\log(3/2)}.$$

$h(t) =$ $100 \cdot 4^{-1/\log(3/2)} t^{1/\log(3/2)}$.

6. [11 points] The graph of $y = F(x)$ is shown below on the left.





a. [4 points] Draw the graph of $y = F^{-1}(x)$ on the provided axes.

b. [4 points] Write a piecewise formula for $F(x)$. (NOT for $F^{-1}(x)$)

Solution: The first segment has slope $\frac{(-2)-(-1)}{1-(-3)} = -\frac{1}{4}$. It passes through the point $(-3,-1)$. Hence, it is a portion of the graph of $y = -\frac{1}{4}(x+3) - 1$. The second segment has slope $\frac{3-1}{5-1} = \frac{1}{2}$. It passes through the point $(1,1)$. Hence, it is a portion of the graph of $y = \frac{1}{2}(x-1) + 1$.

$$F(x) = \begin{cases} -\frac{1}{4}(x+3) - 1 & \text{for } -3 \le x < 1 \\ \frac{1}{2}(x-1) + 1 & \text{for } 1 \le x \le 5 \end{cases}$$

c. [3 points] Let $G(w) = \sqrt{w+1}$ with domain $(0, 8)$, and let $H(w) = F(G(w))$. What is the range for $H(w)$ and $H^{-1}(y)$? Express your answers using **interval notation**.

The range for $H(w)$ is $(1,2)$.

The range for $H^{-1}(y)$ is $(0,8)$.

7. [15 points] Word of Chuck and Samsa's heroics soon spread far and wide. Residents of Chickenville realized that the giant cockroaches were actually very amiable creatures, and began to keep them as pets. In Chickenville t months after February 1,

- The total number of chickens is $P(t)$.
- The total number of pet cockroaches is $C(t)$.
- The total number of wild cockroaches is $W(t)$.

Assume that all of these functions are **increasing**, and assume all cockroaches are either wild or pets. Write mathematical expressions for the following quantities, **including relevant units.**

a. [3 points] The fraction of cockroaches that are kept as pets t months after February 1.

Solution:

$$\frac{C(t)}{C(t)+W(t)}$$

b. [3 points] The average rate of change for the total number of cockroaches between April 1 and June 1.

Solution:

$$\frac{C(4)+W(4)-C(2)-W(2)}{2}$$

c. [3 points] The number of months it takes for the number of pet cockroaches to increase from 62 to 130.

Solution:

$$C^{-1}(130)-C^{-1}(62)$$

d. [3 points] The number of **years** it takes, after February 1, for the number of wild cockroaches to increase to 205.

Solution:

$$\frac{W^{-1}(205)}{12}$$

e. [3 points] Write a practical interpretation of the quantity $P(C^{-1}(327))$.

Solution: It is the number of chickens there are in Chickenville when there are 327 pet cockroaches.

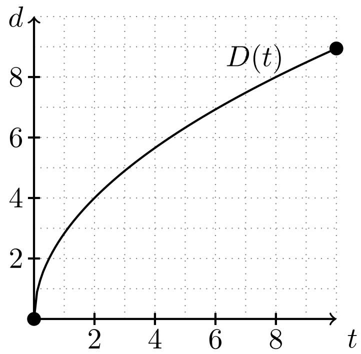
8. [5 points] Let $y = q(x) = \frac{ax}{1+ax}$, where $a > 0$ is a positive constant. Find a formula for the function $q^{-1}(y)$, showing **all** your work.

Solution:

$$y = \frac{ax}{1+ax}$$
$$y(1+ax) = ax$$
$$y = ax(1-y)$$
$$\frac{y}{1-y} = ax$$
$$x = \frac{y}{a(1-y)}$$

$q^{-1}(y) =$ $\frac{y}{a(1-y)}$.

9. [5 points] Consider the function $D(t)$ with its graph shown below on the left, and the piecewise-defined function $S(d)$.



$$S(d) = \begin{cases} 0 & d < 0 \\ -d^2 - 10d + 100 & 0 \le d \le 10 \\ 0 & d > 10 \end{cases}$$

a. [1 point] Is $S(D(t))$ invertible?

[yes] no not possible to tell

b. [4 points] Find all solutions t to the equation $S(D(t)) = 25$. Be sure to show **all** your work and, if necessary, estimate any coordinates on the graph of $D(t)$ to one decimal place.

Solution: We first solve the equation $S(d) = 25$:

$$25 = -d^2 - 10d + 100$$
$$0 = d^2 + 10d - 75 = (d-5)(d+15)$$
$$d = 5,\ 15.$$

Looking at the graph of $D(t)$, the only value of t for which $D(t) = 5$ or 25 is $t = 3.1$.

$t =$ 3.1.

10. [10 points]

a. [2 points] Which of the following functions dominates all the others as $x \to \infty$? **Circle exactly one of the options below.**

$5\left(\frac{2}{3}\right)^x$ $\boxed{e^{0.5x}}$ $(1.6)^{x-3}$ $8(1.6)^{-x}$ $10x^{100}$ $2^{x/2}$

b. [2 points] Which of the following functions dominates all the others as $x \to -\infty$? **Circle exactly one of the options below.**

$5\left(\frac{2}{3}\right)^x$ $e^{0.5x}$ $(1.6)^{x-3}$ $\boxed{8(1.6)^{-x}}$ $10x^{100}$ $2^{x/2}$

c. [2 points] Circle **all** intervals over which $(x-1)^{2016}(x-2)^{2017}(x-3)^{2018}$ is positive.

$(-\infty, 1)$ $(1, 2)$ $\boxed{(2, 3)}$ $\boxed{(3, \infty)}$ NONE OF THESE

d. [2 points] Which of the following functions are periodic? Circle **all** correct options.

$\boxed{e^{\sin(x)}}$ $e^{0.1x}\sin(3x)$ $\cos(x^2)$

$\boxed{\sin^2(2x) + 3\cos^5(4x)}$ NONE OF THESE

e. [2 points] Which of the following expressions could be a formula for $f(x)$, given that $\lim_{x\to\infty} f(x) = \infty$. Circle **all** correct options.

$\boxed{e^{0.01x^2-x}}$ $\boxed{x^6 + e^{-2x}}$ x^6e^{-2x}

$\ln(x+2017) - \ln(x+2016)$ NONE OF THESE