Math 105 — Second Midterm

November 13, 2018

UMID: EXAM SOLUTIONS Initials: __________

Instructor: __________ Section: __________

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1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all phones and smartwatches**, and remove all headphones and earbuds.

Problem	Points	Score
1	11	
2	12	
3	13	
4	9	
5	11	
6	14	
7	17	
8	13	
Total	100	

1. [11 points] On April 22, 1994, the museum where Ross worked received a prehistoric cave painting, and a team of scientists tried to determine its age. The painting contains Carbon-14, but only 15% of the original amount of Carbon-14 was left. The team knew that Carbon-14 decays at a *non-continuous* rate of 1.2% each century (100 years). Let $G(c)$ be the amount of Carbon-14, in grams, left in the painting c centuries after April 22, 1994.
(Note that negative values of c correspond to dates prior to April 22, 1994.)

a. [4 points] If a is the amount of Carbon-14, in grams, the painting contained on April 22, 1994, write a formula for the function $G(c)$. (*Your answer should involve a.)*

Solution: $G(c) = a(0.988)^c$

b. [3 points] What is the continuous decay rate of the function $G(c)$? Give your answer in **exact** form.

Solution:

$$\begin{aligned} e^k &= 0.988 \\ \ln(e^k) &= \ln(0.988) \\ k\ln(e) &= \ln(0.988) \\ k &= \ln(0.988) \end{aligned}$$

c. [4 points] How many centuries before April 22, 1994 was the painting created? Give your answer in **exact** form or estimate it accurately to three decimal places.

Solution: The amount of Carbon-14 when it was first created was $\frac{a}{0.15}$ grams.

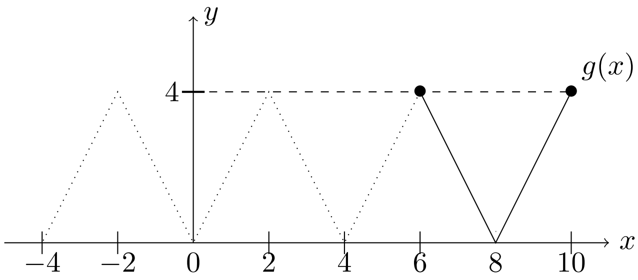
$$\begin{aligned} \frac{a}{0.15} &= a(0.988)^c \\ \ln\left(\frac{1}{0.15}\right) &= \ln(0.988^c) \\ -\ln(0.15) &= c\ln(0.988) \\ c &= -\frac{\ln(0.15)}{\ln(0.988)} \approx -157.14286 \end{aligned}$$

The painting was created $\frac{\ln(0.15)}{\ln(0.988)}$ *or 157.143 centuries before April 22, 1994.*

2. [12 points] A table for the function $f(x)$ and part of the graph of the piecewise-linear function $g(x)$ are given below. The following are true:

- $f(x)$ and $g(x)$ are both defined on $(-\infty, +\infty)$.
- f is an odd function.
- g is a periodic function with period 4.

x	-2	-1	0	2	3
$f(x)$	4	2	a	b	7



a. [3 points] Find the values of a and b.

Solution: Since f is odd, the following holds for every x in $(-\infty, +\infty)$: $f(-x) = -f(x)$. Therefore:
$f(2) = -f(-2) = -4$ and $f(0) = -f(0)$, which means that $f(0) = 0$.

b. [3 points] Find a formula for $g(x)$ for x in $[2, 4]$.

Solution: Since g is a periodic function with period 4, we know that $g(2) = g(6) = 4$ and $g(4) = g(8) = 0$.
The function g is also piecewise linear. Therefore, the slope is : $\dfrac{0-4}{4-2} = -2$ and by the point-slope formula we get: $g(x) = -2(x-2) + 4$.

$g(x) = \underline{\quad -2(x-2)+4 \quad}$ for x in $[2, 4]$.

c. [6 points] Compare the following values by writing one of the symbols: " $<$ ", " $>$ " or " $=$ " in the blank. If the relationship cannot be determined using the information given, write "N" in the blank.

i. [2 points] $g(f(-1))\underline{\quad = \quad}\ g(f(1))$

ii. [2 points] $g(14)\underline{\quad > \quad}\ g(5)$

iii. [2 points] $f(g(3))\underline{\quad = \quad}\ f(g(-3))$

3. [13 points] Solve the following equations for the indicated variable. Your answers should be given in **exact** form. Show carefully **all** your work.

a. [4 points] $\log(x^5) = \pi$. Solve for x.

Solution:

$$\begin{aligned} \log(x^5) &= \pi \\ x^5 &= 10^\pi \\ x &= \sqrt[5]{10^\pi} \end{aligned}$$

b. [4 points] $(\ln(w+4))^3 = e$. Solve for w.

Solution:

$$\begin{aligned} (\ln(w+4))^3 &= e \\ \ln(w+4) &= e^{1/3} \\ w+4 &= e^{(e^{1/3})} \\ w &= e^{(e^{1/3})} - 4 \end{aligned}$$

c. [5 points] $e^{-2p+7} = 10 \cdot 3^p$. Solve for p.

Solution:

$$\begin{aligned} e^{-2p+7} &= 10 \cdot 3^p \\ \ln(e^{-2p+7}) &= \ln(10 \cdot 3^p) \\ (-2p+7)\ln(e) &= \ln(10) + \ln(3^p) \\ -2p+7 &= \ln(10) + p\ln(3) \\ p(\ln(3)+2) &= 7 - \ln(10) \\ p &= \frac{7-\ln(10)}{\ln(3)+2} \end{aligned}$$

4. [9 points] Monica and Rachel recently watched a documentary about earthquakes. They ended up learning a lot about the Richter scale for the magnitude of earthquakes. The Richter rating, R, which measures the magnitude of an earthquake with seismic wave amplitude A is defined as:

$$R = \log\left(\frac{A}{A_0}\right)$$

where A_0 (a positive constant) is the amplitude of the smallest detectable seismic wave. They decide to do some math to get more familiar with the scale. Monica asks Rachel to compute the following:

a. [3 points] An earthquake with a Richter rating of 6 occurred on September 7, 1999 in Athens, Greece. What was the amplitude of its seismic wave? Give your answer in **exact** form. Your answer should be given in terms of A_0.

Solution:

$$\begin{aligned} 6 &= \log\left(\frac{A}{A_0}\right) \\ 10^6 &= \frac{A}{A_0} \\ A &= 10^6 A_0 \end{aligned}$$

Rachel proposes an alternative formula, G, called the "Green rating". The "Green rating" is defined as:

$$G = \frac{1}{2}\left[\log\left(\frac{A}{A_0}\right) + \log\left(\frac{A}{B_0}\right)\right]$$

where B_0 is a constant representing the fixed amplitude of a different seismic wave. Assume that $B_0 > A_0 > 0$.

b. [6 points] Monica notices that the "Green Rating" G is a vertical shift of the original Richter rating R. Determine the shift in terms of A_0 and B_0. Circle UPWARD if it is an upward shift or DOWNWARD if it is a downward shift.

Solution:

$$\begin{aligned} G - R &= \frac{1}{2}\left[\log\left(\frac{A}{A_0}\right) + \log\left(\frac{A}{B_0}\right)\right] - \log\left(\frac{A}{A_0}\right) \\ &= \frac{1}{2}\log\left(\frac{A}{A_0}\right) + \frac{1}{2}\log\left(\frac{A}{B_0}\right) - (\log(A) - \log(A_0)) \\ &= \frac{1}{2}\log(A) - \frac{1}{2}\log(A_0) + \frac{1}{2}\log(A) - \frac{1}{2}\log(B_0) - \log(A) + \log(A_0) \\ &= \frac{1}{2}\log(A_0) - \frac{1}{2}\log(B_0) \\ &= \frac{1}{2}\log\left(\frac{A_0}{B_0}\right) \qquad \text{Since } B_0 > A_0 > 0, \text{ then } \frac{1}{2}\log\left(\frac{A_0}{B_0}\right) < 0 \end{aligned}$$

The vertical shift is UPWARD / $\boxed{\text{DOWNWARD}}$ by $-\frac{1}{2}\log\left(\frac{A_0}{B_0}\right)$ units.

5. [11 points] For each of the following statements, circle the correct answer. **Only one** correct answer is given for each statement. You do not need to show any work for this problem.

a. [2 points] A circle is centered at the point $(3, -1)$ and has radius 2. Starting at the point $(5, -1)$ on the circle, after rotating counter-clockwise by the angle α, the y-coordinate of the corresponding point on the circle must be:

$2\cos(\alpha) - 1$ $2\cos(\alpha) + 3$ [✓ $2\sin(\alpha) - 1$] NONE OF THESE

b. [2 points] If the **continuous** annual growth rate of an exponential function is 40%, then the non-continuous annual growth rate is:

40% $100(e^{0.6}-1)\%$ $e^{0.4}\%$ [✓ NONE OF THESE]

c. [2 points] If θ is any angle given in radians, then $\cos(\theta + \pi)$ must be equal to:

$\cos\theta$ $\sin(-\theta)$ [✓ $-\cos(\theta)$] NONE OF THESE

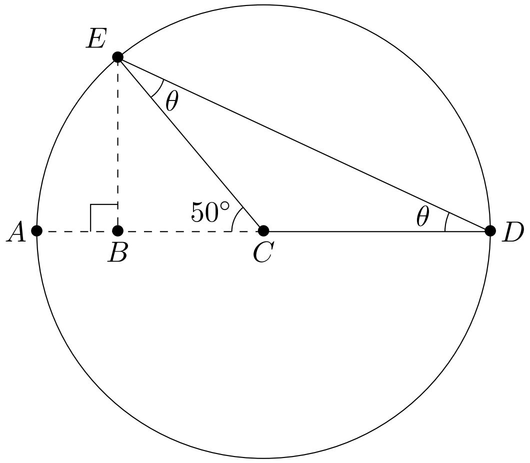
d. [2 points] Let $f(w)$ be a non-constant function with domain $(-\infty, +\infty)$ that satisfies $f(w) + f(-w) = 1$ for all w in $(-\infty, +\infty)$. Then $g(w) = \frac{1}{2} - f(w)$ must be:

[✓ odd] even neither odd nor even CANNOT BE DETERMINED

e. [3 points] If $k(w) = A\sin(w) - 3$ is a periodic function with amplitude 2, then $k(\frac{\pi}{2})$ must be equal to:

0 -1 1 -5 [✓ CANNOT BE DETERMINED]

6. [14 points] Shown below is a circle of **diameter** 10 cm with center C.



Note that the line segment EB is perpendicular to the diameter AD.
Fill in the blanks in the following sentences. Give each answer in **exact** form using only numbers and/or trigonometric functions.

a. [2 points] $\theta =$ $\underline{\quad 25^\circ \quad}$ (degrees).

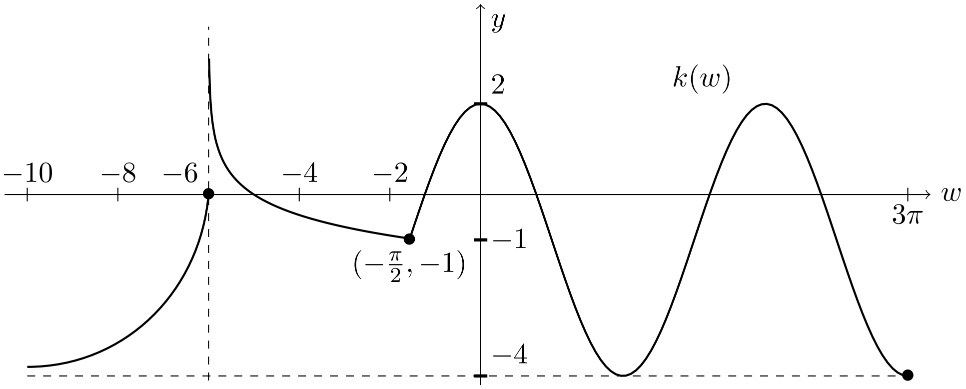
b. [3 points] The length of the arc EA is $\underline{\quad \frac{25}{18}\pi \quad}$ cm.

c. [3 points] The length of the line segment BC is $\underline{\quad 5\cos(50^\circ) \text{ or } 5\sin(40^\circ) \quad}$ cm.

d. [2 points] The length of the line segment AB is $\underline{\quad 5 - 5\cos(50^\circ) \quad}$ cm.

e. [4 points] The length of the line segment DE is $\underline{\quad \frac{5+5\cos(50^\circ)}{\cos(25^\circ)} \text{ or } \sqrt{(5\sin(50^\circ))^2 + (5+5\cos(50^\circ))^2} \quad}$ cm.

7. [17 points] The function $k(w)$ has domain $(-\infty, 3\pi]$. The graph of $k(w)$ for $-10 \leq w \leq 3\pi$ is shown in the picture below:



Assume that the behavior of the graph for w in $(-\infty, -10)$ continues as shown. Moreover, the following are true for the function $k(w)$:

- $\lim_{w \to -6^+} k(w) = +\infty$
- $k(w)$ has a horizontal asymptote $y = -4$.
- $k(w) = A\cos(w) + c$, for $-\pi/2 \leq w \leq 3\pi$.

a. [4 points] Find the values of A and c.

Solution:
$A = 3$ and $c = -1$.

b. [9 points] Fill in the blanks in the following sentences. You can use either interval notation or inequalities, wherever it is needed:

i. The domain of the function $k(-\frac{1}{4}(w-4))$ is $[-12\pi + 4, +\infty)$.

ii. $\lim_{w \to +\infty} -3k(-w) + 1 =$ 13.

iii. The vertical asymptote of the graph of $k(2018w + 2019)$ is $w = -\frac{2025}{2018}$.

c. [4 points] Let $g(w) = -k(5w) - 1.5$. Find the coordinates of the point on the graph of k that correspond to the point $(\frac{2\pi}{5}, -3.5)$ on the graph of g.

Solution: $g(\frac{2\pi}{5}) = -3.5$ leads to:

$$-k\left(5\frac{2\pi}{5}\right) - 1.5 = -3.5$$
$$-k(2\pi) = -2$$
$$k(2\pi) = 2$$

The point on the graph of k is (2π , 2).

8. [13 points] Phoebe, Joey and Chandler are out shopping for their Thanksgiving dinner. They need to buy potatoes and cranberries and have 50 dollars in total. Phoebe will make mashed potatoes with peas and onions, Joey will make tater tots and Chandler will make his famous cranberry sauce. Let:

- $P(a)$ be the cost, in dollars, of buying a **pounds** of potatoes and
- $C(b)$ be the cost in dollars, of buying b **kg** of cranberries.

Also, assume P^{-1} and C^{-1} exist. Each answer for this problem may involve the functions P, C, and/or their inverses.

Write a mathematical expression for the following:

a. [3 points] The amount of cranberries in **kg** that costs 3 dollars.

Solution: $C^{-1}(3)$

b. [4 points] The amount of money, in dollars, that Chandler, Phoebe and Joey have left after buying m **pounds** of potatoes and r **pounds** of cranberries.
Note: 1 **kg** = 2.2 **pounds**.

Solution: $50 - P(m) - C(\frac{r}{2.2})$

Write an equation for the following sentence:

c. [3 points] Buying 1.8 **pounds** of potatoes is 4 dollars more expensive than buying 0.5 **kg** of cranberries.

Solution: $P(1.8) = C(0.5) + 4$

Phoebe, Joey and Chandler end up buying 10 **pounds** of potatoes. Phoebe uses k **pounds** for her mashed potatoes recipe and Joey uses the rest to make tater tots. Answer the following question using a mathematical expression:

d. [3 points] What fraction of the total number of potatoes does Joey use?

Solution: $\dfrac{10-k}{10}$