Math 105 — Second Midterm

November 12, 2019

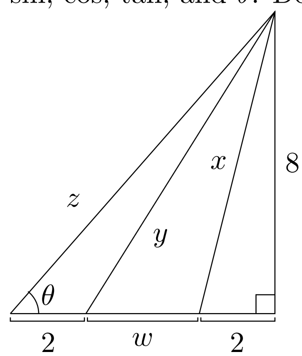
UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. Any work that is erased or crossed out will not be graded.
10. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	9	
2	6	
3	16	
4	14	
5	10	
6	10	
7	10	
8	8	
9	8	
10	9	
Total	100	

1. [9 points] Use the following diagram to answer the questions for this problem. Give your answers in **exact** form in terms of sin, cos, tan, and θ. Do not assume θ is a specific value.



a. [2 points] Find the length of x.

$x =$ $\sqrt{2^2+8^2} = \sqrt{68}$

b. [2 points] Find the length of z.

$z =$ $\frac{8}{\sin(\theta)}$

c. [3 points] Find the length of w.

$w =$ $\frac{8}{\tan(\theta)} - 4$

d. [2 points] Find the length of y in terms of w.

$y =$ $\sqrt{(w+2)^2+8^2}$

2. [6 points] Determine whether the following functions are even, odd, or neither even nor odd. Circle your answer. You do no need to show any work for this problem.

a. [2 points] The function $x^2 + x + 1$ is

EVEN ODD [NEITHER]

b. [2 points] The function $\dfrac{x^4+1}{x^3-x}$ is

EVEN [ODD] NEITHER

c. [2 points] The function $3x\sin(x)$ is

[EVEN] ODD NEITHER

3. [16 points] For each question below, give your answer(s) in **exact** form where appropriate. The different parts of this problem are not related to each other. **Circle** your final answer for each part.

a. [4 points] The point $(3, 7)$ is on the graph of $g(x)$. What point must be on the graph of $-3g(2x-4)$?

Solution: Putting it into the form $-3g(2(x-2))$, we have that the we do a horizontal compression by a factor of $\frac{1}{2}$, followed by a shift right by 2. The vertical transformations are a vertical stretch by a factor of 3, followed by a reflection across the x-axis. Thus, the point we get is

$$(\frac{3}{2}+2, (-3)(7)) = (\frac{7}{2}, -21)$$

b. [4 points] Find all solutions for x:

$$\ln(x^2+e^2) = 3$$

Solution: Exponentiating both sides, we have

$$x^2+e^2 = e^3$$

so

$$x = \pm\sqrt{e^3-e^2}$$

c. [4 points] Find the **tripling** time of the exponential function $f(t) = 120e^{0.7t}$, where t is in hours.

Solution: Solving for tripling time is solving

$$3(120) = 120e^{0.7t}$$

From this, we get

$$\ln(3) = 0.7t$$

$$t = \frac{\ln(3)}{0.7} \text{ hours}$$

d. [4 points] Suppose a farmer can typically grow $B(A)$ bushels of corn on A **acres** of farmland. She starts using a new fertilizer that **doubles** the number of bushels of corn she can grow. Write an expression involving the function B that expresses the number of bushels of corn she can grow on R square meters of farmland if she uses the new fertilizer. (Hint: There are 4046.86 square meters in one acre.)

Solution: Doubling bushel output means we do a vertical stretch by a factor of 2. To get from acres to square meters, we need to horizontally stretch by a factor of 4046.86. Thus, our expression is

$$2B\left(\frac{R}{4046.86}\right)$$

4. [14 points] Roy, the cryptozoologist, has found two more exotic animals on a secret, hidden island! He records the following in his journal:

- **Sorcerer Penguins**: At the time of discovery the population is 3700 penguins, with a continuous yearly decay rate of 4.5%
- **Solar Bears**: $B(t) = 525(0.7)^{2t}$ models the number of solar bears t years after the time of discovery.

Thoughout this problem, give your answers in **exact** form.

a. [3 points] Find a formula for a function $P(t)$ that models the number of Sorcerer Penguins on the island t years after Roy's discovery.

$P(t) =$ $3700e^{-.045t}$

b. [2 points] What is the yearly growth factor of the Sorcerer Penguins?

Yearly growth factor = $e^{-.045}$

c. [3 points] What is the continuous yearly percent decay rate of Solar Bears?

Continuous decay rate = $100(2\ln(0.7))$ %

d. [6 points] Roy also found a third animal on the island, the elusive Iron Lion. The number of Iron Lions t years after Roy's discovery is modeled by the formula

$$I(t) = 50e^{0.5t}$$

Though the initial population of the Iron Lion is much lower, he predicts that they will eventually outnumber the Solar Bears on the island. According to his model, when would that happen? Show all the steps of your calculation, and circle your final answer.

Solution: Setting the two equations equal to each other gives

$$525(0.7)^{2t} = 50e^{0.5t}$$

Taking ln of both sides, we get

$$\ln(525(0.7)^{2t}) = \ln(50e^{0.5t})$$

Applying log rules, we can further reduce to have

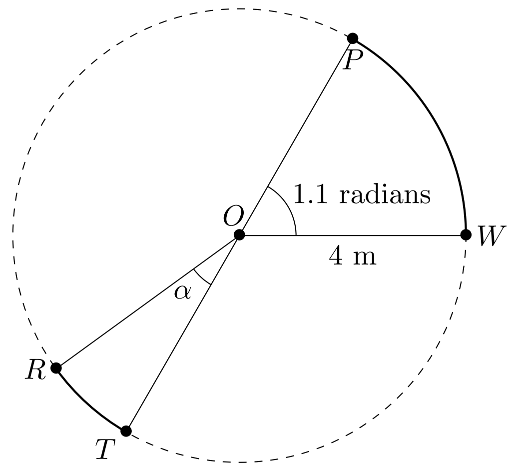
$$\ln(525) + 2t\ln((0.7)) = \ln(50) + 0.5t$$

We can then solve directly for t, getting

$$\ln(525) - \ln(50) = (0.5 - 2\ln(0.7))t$$

$$t = \frac{\ln(525) - \ln(50)}{0.5 - 2\ln(0.7)}$$

5. [10 points] Use the diagram below to answer the following questions. The diagrams contains a circle of radius 4 with center point O at coordinates $(0,0)$. All angles in the figure are given in radians, and all lengths are in meters.



In all parts of this problem, give your answers in **exact** form and include units. It is okay to leave your answers in terms of the sin, cos, and tan functions. You don't need to show work on this problem, but you could receive credit for correct work shown.

a. [3 points]
Convert 1.1 radians into **degrees**.

1.1 radians is equivalent to $\frac{1.1(180)}{\pi}^{\circ}$

b. [3 points]
Find the coordinates of the point P.

The coordinates of P are $(4\cos(1.1), 4\sin(1.1))$

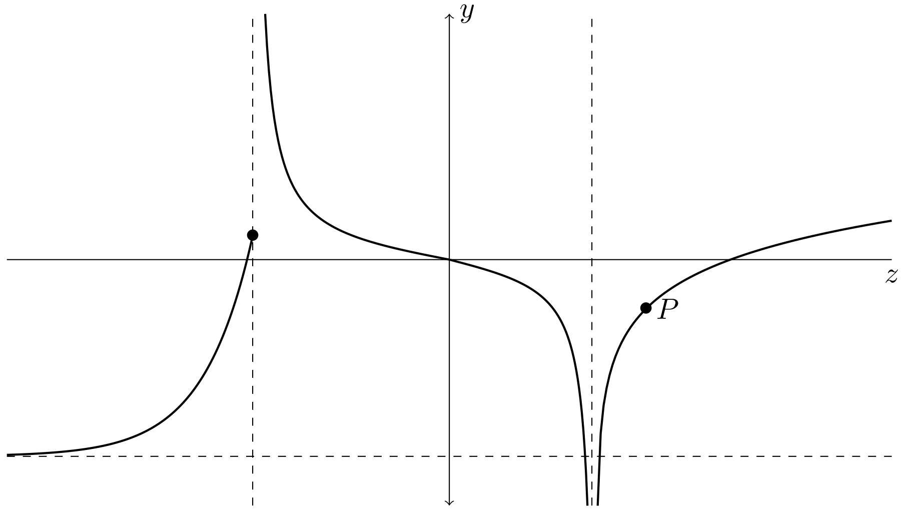
c. [2 points]
Find the length of the bold arc between W and P.

The length of the arc is (1.1)(4) meters

d. [2 points]
The length of the bold arc between R and T is 1.2 meters. Find the angle α in radians.

$\alpha =$ $\frac{1.2}{4}$

6. [10 points] Below is a graph of the function $S(z)$. The function has a horizontal asymptote at $y = -4$, and vertical asymptotes at $z = -4$ and $z = 3$. The point P is located at the coordinates $(4, -1)$.



a. [4 points]
For $z > 3$, the formula for $S(z)$ is of the form $\log(z-h)+k$. In exact form, find the values of h and k using the fact that $P = (4, -1)$ and the fact that $z = 3$ is a vertical asymptote of $S(z)$.

Solution: $\log(z)$ usually has an asymptote at $z = 0$, so for the asymptote to appear at $z = 3$, we need a horizontal shift to the right by 3 units. This gives us that $h = 3$. With this, we can plug in the point P to get

$$\begin{aligned} -1 &= \log(4-3) + k \\ &= \log(1) + k \\ &= k \end{aligned}$$

$h =$ 3

$k =$ -1

Let $T(z) = 3S(-0.5(z-3)) - 8$.
Note: The next two parts of this problem are about $T(z)$, not about the original function!

b. [4 points] Find the vertical asymptote(s) of $T(z)$. Circle your answer(s).

Solution: The vertical asymptotes of $T(z)$ only depend on the horizontal transformations applied to $S(z)$. We see that the transformations applied are a reflection about the y-axis, followed by a horizontal stretch by a factor of 2, and then a horizontal shift to the right by 3. Applying these to the vertical asymptotes, we get

$$z = 2(-3) + 3 = -3$$

$$z = 2(-(-4)) + 3 = 11$$

c. [2 points] Find $\lim_{z\to\infty} T(z)$.

Solution: Note that

$$\lim_{z\to\infty} T(z) = \lim_{z\to-\infty} 3S(-0.5(z-3)) - 8 = 3(\lim_{z\to-\infty} S(z)) - 8$$

Since $\lim_{z\to-\infty} S(z) = -4$, we have

$$\lim_{z\to\infty} T(z) = 3(-4) - 8 = -20$$

$\lim_{z\to\infty} T(z) =$ -20

7. [10 points] Let $P(r)$ be a periodic function, defined for all real numbers r, where

- $P(r)$ has period 8
- $P(r)$ has midline $y = 4$
- $P(r)$ has amplitude 6.
- $P(r)$ attains its minimum value at $r = 5$.

a. [4 points] Fill in each blank with an appropriate value in the following table using the information about $P(r)$ given above.

r	-5	4	5	12
$P(r)$	7	6	-2	6

Solution: $P(5)$ is the minimum value, which is $4 - 6 = -2$ using the midline and amplitude. $P(12) = P(4) = 6$ by using that the period is 8.

b. [2 points] What is the value of $P(2019)$?
If it's not possible to find the value, write "NOT POSSIBLE." Circle your final answer.

Solution: We have that 2019/8 has a remainder of 3, so using periodicity with period 8, we get

$$P(2019) = P(3) = P(-5) = 7$$

c. [1 point] What is the maximum value attained by $P(r)$?
If it's not possible to find the value, write "NOT POSSIBLE." Circle your final answer.

Solution: The max value is $4+6 = 10$, obtained by looking at the midline and amplitude of the function.

d. [3 points] Can you tell for sure at which r-coordinates $P(r)$ attains its maximum? If so, give one such value and briefly explain your answer. If not, briefly explain why.

Solution: No. A function being periodic doesn't imply anything about where the maximum could occur. Since we don't know the general shape of the function, we cannot determine where the maximum is.

8. [8 points] Archaeologists have discovered what seems to be scientific research papers near some dinosaur fossils. The papers talk about the "danger level", L, of a potential asteroid impact. From what they can read, the formula is given by

$$L = 3\log\left(\frac{4M}{k}\right)$$

where M is the mass of the asteroid, in kg, and k is a positive constant. For this problem, leave all your answers in **exact** form.

a. [4 points]
Suppose an asteroid has a danger level of 7.5. What would the mass of the asteroid be? Your answer should include units, and may involve the constant k.

Solution:

$$\begin{aligned} 10^{7.5} &= 10^{3\log\left(\frac{4M}{k}\right)} \\ &= 10^{\log\left(\left(\frac{4M}{k}\right)^3\right)} \\ &= \left(\frac{4M}{k}\right)^3 \end{aligned}$$

Solving then gives $M = \frac{k}{4}(10^{7.5/3})$

Mass = $\frac{k}{4}(10^{7.5/3})$

b. [4 points]
Let N be the danger level of an asteroid of mass $12A$ kg, and let n be the danger level of an asteroid of mass $5A$ kg, where A is a positive constant.
Compute $N - n$. Simplify your answer so that it does *not* include k or A.

Solution: We have $N = 3\log\left(\frac{4(12A)}{k}\right)$ and $n = 3\log\left(\frac{4(5A)}{k}\right)$. Setting up the difference, we get

$$\begin{aligned} N - n &= 3\log\left(\frac{4(12A)}{k}\right) - 3\log\left(\frac{4(5A)}{k}\right) \\ &= 3(\log\left(\frac{4(12A)}{k}\right) - \log\left(\frac{4(5A)}{k}\right)) \\ &= 3(\log\left(\frac{48A}{k}\cdot\frac{k}{20A}\right)) \\ &= 3\log\left(\frac{48}{20}\right) \end{aligned}$$

Where we used a log rule in the third line.

$N - n =$ $3\log\left(\frac{12}{5}\right)$

9. [8 points] Let $h(x) = \frac{4}{3}g(3(x+5)) - 9$. Write out in words a sequence of transformations that, when applied to the graph of $h(x)$ result in the graph of $g(x)$.

Note: You are transforming the graph of $h(x)$ to the graph of $g(x)$ here, and **not** the other way around.

In the first blank on each line, write one of the transformations from the list at the end of the problem. In the second blank, write a number that represents the appropriate shift or scaling factor. If you don't need to use all the lines below to write out the transformation, leave any remaining lines blank.

First, Shift up by 9.

Then, Compress vertically by $\frac{3}{4}$.

Then, Shift to the right by 5.

Then, Stretch horizontally by 3.

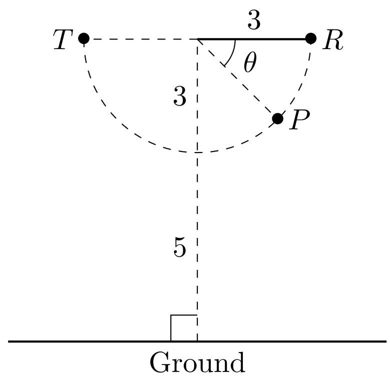
List of transformations to choose from for the first blank on each line above:

Shift to the left	Shift up	Stretch vertically	Stretch horizontally
Shift to the right	Shift down	Compress vertically	Compress horizontally

Solution: Note that other orders are possible, as long as they give the same resulting transformation.

10. [9 points] A pendulum is swinging in a semi-circular arc of radius 3 feet pictured below. The pendulum starts at the point R and swings along the arc until it reaches the point T. Then, it swings back to the point R along the arc. The motion then repeats.

Assume that the line through the points T and R is parallel to the ground.



a. [4 points] Suppose $h(t)$ is the height of the pendulum above the ground t seconds after it is at the point R. Find the amplitude and midline of the graph of $y = h(t)$.

Solution: The maximum and minimum heights reached by the pendulum are 8 feet and 5 feet above the ground respectively. We can then compute the amplitude as $\frac{8-5}{2} = \frac{3}{2}$, and the midline as $y = \frac{8+5}{2} = \frac{13}{2}$.

Amplitude: $\frac{3}{2}$ Midline: $y = \frac{13}{2}$

b. [2 points] The function $h(t)$ defined in part (a) has period 4. Find the period of the function $3h(5t)$.

Solution: The period is affected by the horizontal compression. Since we compress by a factor of $\frac{1}{5}$, the new period is $\frac{4}{5}$.

Period of $3h(5t)$: $\frac{4}{5}$

c. [3 points] The angle θ measures $\frac{3\pi}{10}$ radians. Find the height of the pendulum above the ground when it is at the point P. Give your answer in **exact** form.

Solution: This is like finding the point on a circle of radius 3 with center at $(0, 8)$. The height corresponding to P would then be $3\sin(-\theta) + 8 = -3\sin(\theta) + 8$.

Height of pendulum at P: $-3\sin(\theta) + 8$