Math 105 — Final Exam

December 17, 2019

UMID: EXAM SOLUTIONS Initials: ____________

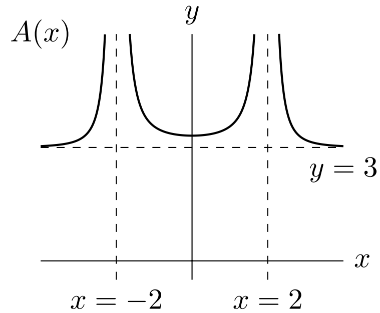
Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 11 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. Any work that is erased or crossed out will not be graded.
10. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	9	
2	11	
3	10	
4	17	
5	4	
6	6	
7	4	
8	5	
9	12	
10	10	
11	12	
Total	100	

1. [9 points]

a. [4 points]
Let $A(x)$ be the function graphed below with end behavior shown. $A(x)$ has vertical asymptotes at $x = -2$ and $x = 2$, and it has a horizontal asymptote at $y = 3$.



Find the domain and range of $\ln(A(x))$. Give your answer in interval notation, using exact form for any numbers in the endpoints of your interval(s).

The domain of $\ln(A(x))$ is $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

The range of $\ln(A(x))$ is $(\ln(3), \infty)$.

b. [5 points]
The table below has some values of the function $B(t)$.

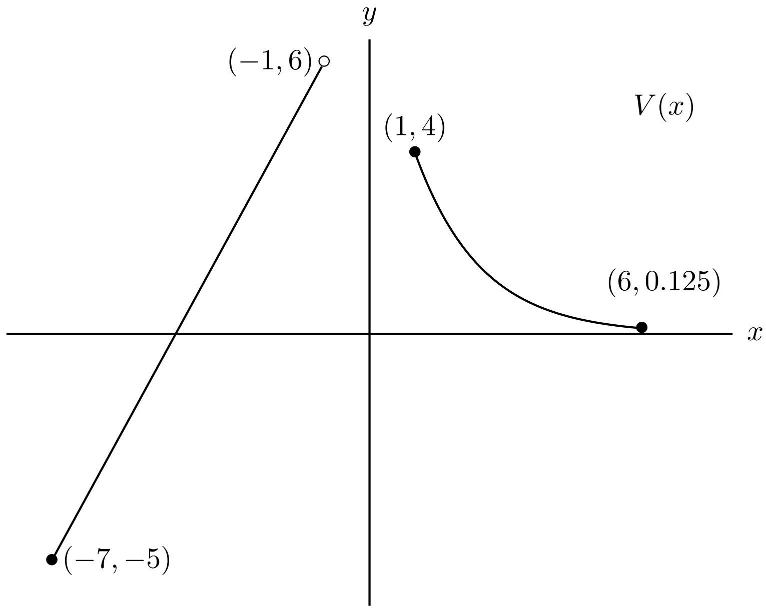
t	0	1	3	7	8	9	11
$B(t)$	3	2	6	4	12	3	1

Find all solutions of the equation

$$(B(t))^2 - 7B(t) + 12 = 0$$

that can be determined using only the information in the table above. Circle your final answer(s).

2. [11 points]
Let $V(x)$ be a function whose graph is pictured below. It has two pieces - one piece is a **linear** function and one is an **exponential** function. Do not assume anything about this function outside of the part shown in the graph below.



Write a piecewise-defined formula for $V(x)$. For this problem you will be graded both on the correctness of your formulas for each piece and on the use of piecewise notation. Circle your final answer for $V(x)$.

Solution: The linear part can be found using the two points given. The slope between them is $\frac{11}{6}$, and with the point $(-7,-5)$, we get that the equation for the linear part is $V(x) = \frac{11}{6}(x+7) - 5$.
As for when $V(x)$ is exponential, we can again find the formula using the two points shown. Assuming the general form of $V(x) = ab^x$, we have that

$$\frac{0.125}{4} = \frac{1}{32} = b^5$$

So that $b = \frac{1}{2}$. Now using the point $(1, 4)$, we get

$$a(\frac{1}{2}) = 4$$

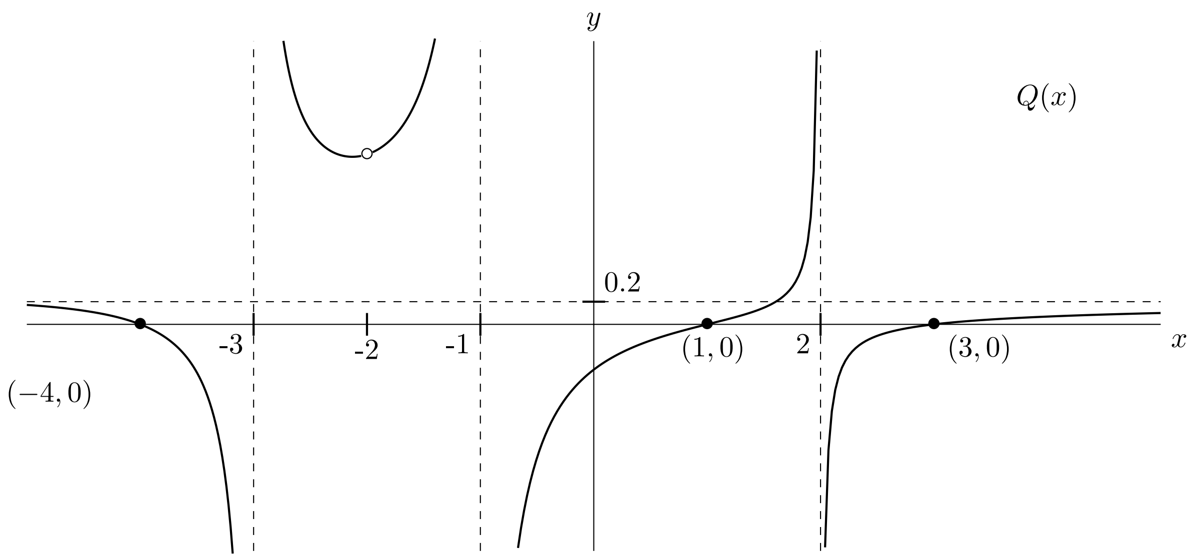
so $a = 8$. Hence, $V(x) = 8(\frac{1}{2})^x$ on the exponential piece.
Putting them together, we get the piecewise function

$$V(x) = \begin{cases} \frac{11}{6}(x+7) - 5 & \text{for } -7 \leq x < -1 \\ 8(\frac{1}{2})^x & \text{for } 1 \leq x \leq 6 \end{cases}$$

3. [10 points]
Below is the graph of a rational function $Q(x)$. Note that

- $Q(x)$ has a horizontal asymptote at $y = 0.2$
- $Q(x)$ has zeros at $x = -4$, $x = 1$, and $x = 3$
- $Q(x)$ has a hole at $x = -2$
- $Q(x)$ has vertical asymptotes at $x = -3$, $x = -1$, and $x = 2$

Using the information in the portion of the graph shown, write a possible formula for $Q(x)$. You do not need to simplify your answer.



Solution: Since there are vertical asymptotes at $x = -3$, $x = -1$ and $x = 2$, we have that we must have $(x+3)(x+1)(x-2)$ in the denominator. Similarly with the zeros, we determine that $(x+4)(x-1)(x-3)$ is in the numerator.
There hole at $x = -2$, but the hole is not a zero. This means that the numerator and the denominator both have equal amounts of $(x+2)$ factors.
Finally, a horizontal asymptote at $y = 0.2$ implies that the degrees of both the numerator and denominator agree, and that the fraction of their leading coefficients is 0.2. Putting it all together, we can construct a possible formula of the form

$$Q(x) = \frac{0.2(x+4)(x-1)(x-3)(x+2)}{(x+3)(x+1)(x-2)(x+2)}$$

$Q(x) =$ $Q(x) = \frac{0.2(x+4)(x-1)(x-3)(x+2)}{(x+3)(x+1)(x-2)(x+2)}$

4. [17 points] The following table contains information about the functions $F(x)$, $G(x)$, and $H(x)$. The functions satisfies the following properties:

- $F(x)$ is a power function.
- $G(x)$ is an odd, periodic function with period 7.
- $H(x)$ is a quadratic function with average rate of change -1.5 on $[0, 5]$

Assume that all three functions are defined for all real numbers.

x	2	5	8
$F(x)$	$\frac{3}{5}$	$\frac{75}{8}$	$\frac{192}{5}$
$G(x)$	4	?	11
$H(x)$	0.4	-3.5	-5.6

a. [8 points]
Compute the following values. If it cannot be determined, write NEI. You don't need to show work on this part of the problem, but you could receive partial credit for work shown.

- $F(0) =$ 0
- $G(0) =$ 0
- $G(1) =$ 11
- $G(5) =$ -4
- $G(-8) =$ -11
- $H(0) =$ 4

b. [6 points]
Find a formula for $F(x)$. Circle your answer.

Solution: Using the table, we can choose two points and find the power function through them. For instance, we can use $x = 2$ and $x = 8$. With the general form $F(x) = kx^n$, for k a constant, we have that

$$\frac{F(5)}{F(2)} = \frac{8^n}{2^n} = 4^n = \frac{\frac{192}{5}}{\frac{3}{5}} = 64$$

We solve for n from this, and get $n = 3$. Then putting it back into the equation for $F(2)$, we get

$$F(2) = \frac{3}{5} = k(2)^3$$

so that $k = \frac{3}{40}$. Hence

$$F(x) = \frac{3}{40}x^3$$

c. [3 points]
What is the sign of the leading coefficient of $H(x)$? Give a brief justification of how you determined it.

Solution: Since $H(x)$ is quadratic, the sign leading coefficient can be determined by its concavity. Computing the average rate of change between $H(2)$ and $H(5)$ gives us -1.3, and the average rate of change between $H(5)$ and $H(8)$ is -0.7. Since the average rate of change is increasing, the quadratic is concave up, hence the sign is positive.

5. [4 points]
Let α and β be constants such that

- $\ln(\alpha) = 2$
- $\ln(\beta) = 5$

Find the value of $\ln(\alpha^6\beta^{-3}e^{25})$. Your answer should **not** include α, β, or ln.

Solution: Using log rules, we can write

$$\begin{aligned}\ln(\alpha^6\beta^{-3}e^{25}) &= \ln(\alpha^6) + \ln(\beta^{-3}) + \ln(e^{25})\\ &= 6\ln(\alpha) - 3\ln(\beta) + 25\\ &= 12 - 15 + 25 = 22\end{aligned}$$

$\ln(\alpha^6\beta^{-3}e^{25}) =$ 22

6. [6 points]
Let $P(x)$ be a polynomial with the following properties:

- $P(x)$ only has zeros at $x = -3, -1, 2$
- $P(x)$ has degree 4
- The graph of $P(x)$ passes through the points $(-4, -36)$ and $(-2, -8)$

Find a formula for $P(x)$. You do not need to simplify your answer.

Solution: $P(x)$ has degree 4 but only has three zeros, so one of them must be a double zero. Note that at $x = -4$ and $x = -2$, the outputs are both negative, yet there is a zero between them. This implies that $x = -3$ must be a double zero, so that our polynomial is of the form

$$P(x) = c(x+3)^2(x+1)(x-2)$$

where c is some constant. To find c, we can use one of the points, say $(-2, -8)$. Plugging it in, we get

$$-8 = c(-2+3)^2(-2+1)(-2-2) = 4c$$

Hence $c = -2$.

$P(x) =$ $-2(x+3)^2(x+1)(x-2)$

7. [4 points]

Find all possible solutions w to the equation $\dfrac{2e^{9(w-1)}}{7} = 3$. Be sure to show all your steps, give your answer in **exact** form, and **circle** your final answer.

Solution: We can multiply both sides by $\frac{7}{2}$ to get

$$e^{9(w-1)} = \frac{21}{2}$$

Taking ln of both sides then gives

$$9(w-1) = \ln(\frac{21}{2}) = \ln(21) - \ln(2)$$

and solving for w, we get

$$w = \frac{\ln(21) - \ln(2))}{9} + 1$$

8. [5 points]

Consider the function

$$R(q) = \frac{5q+3}{4-q}$$

Find a formula for its inverse. Be sure to show all your steps, and **circle** your final answer.

Solution: If we let $W = R(q)$, then we can set

$$W = \frac{5q+3}{4-q}$$

Solving for q in terms of W, we get

$$W(4-q) = 5q+3$$

Putting all the q terms to the same side gives

$$5q + Wq = 4W - 3$$

And using that $5q + Wq = q(5+W)$, we divide to get

$$q = \frac{4W-3}{5+W}$$

Hence our function for the inverse should be

$$R^{-1}(W) = \frac{4W-3}{5+W}$$

9. [12 points]

a. [6 points]
While searching for cryptids, Roy claims he found a secret island with crazy thermodynamic properties. According to him, the temperature on the island fluctuates in a 24 hour cycle that can be modeled by a sinusoidal function. The maximum temperature of 45° Celsius occurs at 1 p.m. every day, and the minimum temperature of −25° Celsius occurs at 1 a.m. every day. Let the sinusoidal function $C(t)$ be the temperature, in degrees Celsius, on the island t hours after 8 a.m. Find a formula for $C(t)$.

Solution: Since it fluctuates in a 24 hour cycle, we have that the period of the function is 24. Furthermore, the midline is $y = \frac{45+(-25)}{2} = 10$ and the amplitude is $\frac{45-(-25)}{2} = 35$. Thus, we have that

$$C(t) = 35\cos(\frac{2\pi}{24}(t-h)) + 10$$

for some shift h. Note that the maximum for our function is at 1 p.m, which is 5 hours after 8 a.m. Since $\cos(t)$ naturally has a maximum at $t = 0$, and we want the maximum to be at $t = 5$, we want to shift 5 to the right. Therefore, we want $h = 5$, giving us

$$C(t) = 35\cos(\frac{2\pi}{24}(t-5)) + 10$$

b. [6 points]
On the island, Roy also claims to have found a population of the elusive Megaconda! In his notes, he writes that it is clear that the population size of Megaconda population must fluctuate in a sinusoidal manner, and that there are $M(t)$ thousand Megacondas t months after his discovery. Let

$$M(t) = 13\sin\left(\frac{\pi t}{3}\right) + 25$$

Find the first two times after Roy's discovery when the Megaconda population is 18,000. Give your answers using **exact** form.

Solution: We set up the equation

$$18 = 13\sin\left(\frac{\pi t}{3}\right) + 25$$

After some algebra, we can write this as

$$\frac{-7}{13} = \sin\left(\frac{\pi t}{3}\right)$$

We can then solve for the principal value by taking $\sin^{-1}$ of both sides, giving us

$$\sin^{-1}(\frac{-7}{13}) = \frac{\pi t}{3}$$

So

$$t = \frac{3\sin^{-1}(\frac{-7}{13})}{\pi}$$

However, this value is negative, and we want the first two positive t-values.
The first positive value can be found from the principal value using symmetry, giving us

$$t = 3 - \frac{3\sin^{-1}(\frac{-7}{13})}{\pi}$$

The second positive value can be found by adding the period to the principal value, giving us

$$t = 6 + \frac{3\sin^{-1}(\frac{-7}{13})}{\pi}$$

From this, we see that the first two times the Megaconda population is $18,000$ is $3 - \frac{3\sin^{-1}(\frac{-7}{13})}{\pi}$ and $6 + \frac{3\sin^{-1}(\frac{-7}{13})}{\pi}$ months after Roy's discovery.

10. [10 points] When not selling cards, Rowena runs a rather popular ice cream shop in town. Her store carries only two flavors, mango and strawberry, which she sells for $M(k)$ and $S(k)$ dollars, respectively, for k kilograms. Assume that both functions are invertible, but **do not** assume anything else about them. Your answers for this problem may involve M, S, or their inverses.

a. [2 points]
Give a practical interpretation of $S^{-1}(4.7)$.

Solution: $S^{-1}(4.7)$ is the amount of kilograms of strawberry ice cream that costs 4.7 dollars.

b. [3 points]
Give a practical interpretation of $M^{-1}(S(1.5)) = 1$.

Solution: One kg of mango ice cream is the same price as 1.5 kg of strawberry ice cream.

c. [2 points]
Write an equation that expresses the following: "7 kg of strawberry ice cream costs 4 dollars less than 5 kg of mango ice cream."

Solution: $S(7) + 4 = M(5)$

d. [3 points]
A customer bought T total kg of ice cream at Rowena's shop. If they spent \$20 on strawberry ice cream, find an expression for the amount, in dollars, they spent on mango ice cream. Your answer may involve T.

Solution: $M(T - S^{-1}(20))$

11. [12 points] For each of the questions below, circle **all** solutions that are correct.

a. [3 points]

Let $Q(x) = \dfrac{(3+2x)(6x^2-9)}{(3x^2+1)(7-x)}$.

What are the **horizontal** asymptote(s) of $2Q(3x+6)+7$?

$\boxed{y=-1}$ $\quad y=3 \quad y=-6 \quad y=-11$

$y=-4 \quad y=\dfrac{12}{7}$ None of these

b. [3 points]

If $\sin(x) = \frac{4}{5}$, then what value(s) can $\cos(x)$ be?

$\boxed{\dfrac{3}{5}} \quad \dfrac{1}{3} \quad \boxed{-\dfrac{3}{5}} \quad \dfrac{\sqrt{3}}{2}$

$-\dfrac{\sqrt{3}}{2} \quad -\dfrac{1}{3}$ None of these

c. [3 points]

The function $f(x)$ has the property $\lim\limits_{x\to\infty} f(x) = \infty$. Which of the following could be $f(x)$?

$\boxed{\ln(x)} \quad \boxed{\dfrac{.001e^x}{30x^{100}+14x^{200}}} \quad e^{\sin(x)+\cos(x)} \quad \dfrac{x^{\frac{1}{2}}+4}{(\ln(x))^4 - x^{\frac{2}{3}}}$

$x^{-2} \quad \dfrac{x^4+3x^2+7}{3x^3+x+x^5}$ None of these

d. [3 points]

Which functions are periodic with period 4?

$\boxed{5\sin(\frac{\pi}{2}(x-3))+1} \quad 4\cos(\frac{2}{\pi}(x+2)) \quad \boxed{\tan(\frac{\pi x}{4})} \quad e^{\cos(4x)}$

$\tan(\frac{\pi x}{2})+4 \quad e^{\sin(\frac{2x}{\pi})}$ None of these