Math 105 — First Midterm — October 20, 2020

EXAM SOLUTIONS

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1. This exam has 9 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Partial credit will be awarded for correct work.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.
5. You do not need to "simplify" your answers unless asked to do so.
6. You must use the methods learned in this course to solve all problems. Logarithm functions taught in this course include "log" (log base 10) and "ln" (natural log).
7. You may use one pre-written page of notes, on an 8.5"x11" standard sheet of paper, with whatever you want written on both sides.
8. You will not be allowed to use any other resources, including calculators, other notes, or the book.

9. You must write your work and answers on **blank, white, physical paper**.
10. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
11. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	2
2	8
3	12
4	9
5	13
6	13

Total	57

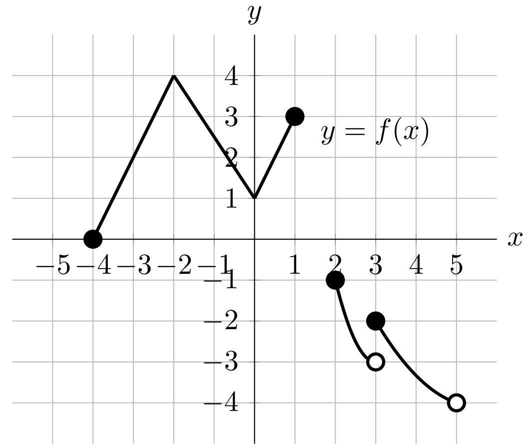
1. [2 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want written on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person other about the exam until **8am on Wednesday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [8 points]

The entire graph of a function $y = f(x)$ is given to the right. Note that $f(x)$ is piecewise-linear for $-4 \leq x \leq 1$.



A different function, $g(x)$, is given in the following table.

x	-3	-2	-1	0	0.2	1	2	3
$g(x)$	5	4	1	0	-1	-2	-4	-3

a. [3 points] Find

(i) the domain of $f(x)$ and

(ii) the range of $f(x)$.

Write your answers using inequalities or interval notation. Make sure to clearly label which answer is the domain and which is the range. You do not need to justify your answer.

Solution: (i) The domain of $f(x)$ is $-4 \leq x \leq 1$ and $2 \leq x < 5$ (or in interval notation: $[-4, 1]$ and $[2, 5)$.)

(ii) The range is $-4 < x \leq -2$ and $0 \leq x \leq 4$ (in interval notation: $(-4, -2]$ and $[0, 4]$.)

b. [5 points] Find the following values. If you make any calculations to find your answers, include those calculations in your submission.

(i) The average rate of change of $f(x)$ from $x = -4$ to $x = 1$

(ii) $g(f(3))$

(iii) All values of x so that $g(f(x)) = -2$

Solution:

(i) The average rate of change of f between $x = -4$ and $x = 1$ is

$$\frac{f(1) - f(-4)}{1 - (-4)} = \frac{3 - 0}{5} = \frac{3}{5}.$$

(ii) Since $f(3) = -2$, we have $g(f(3)) = g(-2) = 4$.

(iii) If the output of g is -2, the input of g must be 1, and so if $g(f(x)) = -2$, then $f(x) = 1$. So, we just need to solve $f(x) = 1$ for x. From the graph, we see there are two values where this happens:

$$x = -3.5 \quad \text{and} \quad x = 0.$$

(To see why $x = -3.5$ is a solution: we know $f(x)$ is linear between $x = -4$ and $x = -2$, and has slope 2. Since $f(-4) = 0$, if we want the output to go up 1, the input needs to increase by .5, yielding $x = -3.5$.)

3. [12 points] A gardener is growing a plant.

- Let t be the number of days after the plant first sprouts.
- The height of the plant t days after it sprouts is $H(t)$ inches.
- The gardener gives the plant $W(t)$ cups of water on the t^{th} day after it sprouts.
- When the gardener uses M cups of water, she mixes in $V(M)$ teaspoons of special plant vitamins.

Suppose that $V(M)$ and $H(t)$ have inverses.
For each of the following, give a practical interpretation of the expression in the context of the problem, or explain why the expression does not make sense in this context.

a. [3 points] $H(6) = 3$

Solution: The height of the plant 6 days after it sprouts is 3 inches.

b. [3 points] $V(W(4))$

Solution: $V(W(4))$ is the number of teaspoons of special plant vitamins the gardener mixed in on the fourth day after the plant sprouted.

c. [3 points] $W(H(6))$

Solution: This expression does not make sense: the units of $H(6)$ are inches, while the input of W should be days.

d. [3 points] $\dfrac{H^{-1}(12) - H^{-1}(9)}{12 - 9} = 2$

Solution: Between when the plant was 9 inches and 12 inches tall, it took the plant an average of 2 days to grow 1 inch taller.

4. [9 points] Let $h(w)$ be a function, with values given in the following table:

w	1	2	4
$h(w)$	8	10	14

a. [2 points] Briefly explain why $h(w)$ could be a linear function of w, and find a linear formula for $h(w)$. For the rest of the problem, assume that $h(w)$ is linear.

Solution: We need to check that the rate of change of $h(w)$ is constant. The two rates of change we can calculate from the table are $\frac{h(2)-h(1)}{2-1} = \frac{10-8}{1} = 2$ and $\frac{h(4)-h(2)}{4-2} = \frac{14-10}{2} = 2$, and so the rate of change is constant and thus $h(w)$ could be linear. Now, we find a formula for $h(w)$. We know already that the rate of change is 2, and this is just the slope. So, we plug in a point (say $(1, 8)$) to find the h-intercept:

$$\begin{aligned} h(w) &= 2w + b \\ 8 &= 2(1) + b \\ 6 &= b. \end{aligned}$$

Thus, we know that

$$h(w) = 2w + 6.$$

b. [3 points] Let $q(w)$ be a quadratic function of w, where $q(w)$ has its vertex at $(1, 8)$ and passes through the w-intercept of $h(w)$. Find a formula for $q(w)$.

Solution: We use vertex form: since the vertex of $q(w)$ is at $(1, 8)$, we can write

$$q(w) = a(w-1)^2 + 8.$$

We just need to find a, and to do so we plug in any other point on the graph of $q(w)$. We're told that the w-intercept of $h(w)$ is on the graph, so let's find that. We set $h(w) = 0$ and solve:

$$\begin{aligned} 2w + 6 &= 0 \\ 2w &= -6 \\ w &= -3. \end{aligned}$$

Thus the point $(-3, 0)$ is on the graph of $q(w)$. Plugging this in:

$$0 = a(-3-1)^2 + 8.$$

Solving for a, we get $a = -1/2$. So, we have

$$q(w) = -\frac{1}{2}(w-1)^2 + 8.$$

c. [1 point] Is the graph of $q(w)$ concave up or concave down?

Solution: Because $a = -1/2 < 0$, the graph is concave down. (You can also note that since the vertex of $q(w)$ is above the horizontal axis, if the graph were concave up it would "open upward" and could have no zeroes.)

d. [3 points] Find the zeroes of $q(w)$.

Solution: We set $q(w) = 0$ and solve for w:

$$\begin{aligned} -\frac{1}{2}(w-1)^2 + 8 &= 0 \\ -\frac{1}{2}(w-1)^2 &= -8 \\ (w-1)^2 &= 16 \\ w - 1 &= \pm 4 \\ w &= 1 \pm 4 \\ w &= -3, 5. \end{aligned}$$

(You can also use symmetry: we know already that $q(w)$ passes through $(-3, 0)$, and is symmetric about the vertical line $x = 1$ passing through its vertex. Since the distance from $x = -3$ to $x = 1$ is 4, the next zero occurs 4 after $x = 1$, i.e., at $x = 5$.)

5. [13 points] You're buying baking supplies (flour, butter, apples, etc.) for your pie business. Let $A(d)$ be the amount of money (in dollars) you'll get back from selling the pies you make from d dollars of supplies. For each dollar you spend on supplies between 0 and 50 dollars, you'll get back \$2. For each additional dollar over the first 50 dollars you invest you get back \$3.

a. [4 points] From the description above, write down a piecewise formula for the function $A(d)$. Use standard piecewise notation:

$$A(d) = \begin{cases} \\ \\ \end{cases}$$

Solution: For the first 50 dollars, you get back 2 dollars for each dollar you invest, so the formula is just $A(d) = 2d$ on that interval. After the first 50, you get 3 dollars for every dollar you invest. So, if you spend $d \geq 50$ dollars, you get:

- 2 dollars for each of the first 50 dollars you spend, or 100 dollars, and
- 3 dollars for each dollar beyond the first 50 dollars, that is, $3(d-50)$ dollars.

So, the piecewise formula is given by

$$A(d) = \begin{cases} 2d & 0 \leq d \leq 50 \\ 3(d-50)+100 & 50 \leq d \end{cases}$$

b. [4 points] Evaluate $A^{-1}(190)$ and give a practical interpretation of your answer.

Solution: To make 190 dollars, it's clear you have to invest more than 50 dollars (as this would only get you 100 dollars). So, you need to invest the first 50 (receiving 100 back), and then you need to make 90 more; since each dollar after the first 50 dollars returns 3, you just need to invest $90/3 = 30$ more dollars, for a total of 80 dollars.
A practical interpretation of the answer is "To get 190 dollars back in sales, you need to invest 80 dollars in supplies".

c. [5 points] Find a piecewise formula for the composition $A(A(d))$. Use standard piecewise notation.

Solution: For $d \leq 50$, we have $A(d) = 2d$, and when $2d \leq 50$ we then have $A(A(d)) = A(2d) = 4d$. This stops being valid when $2d \geq 50$, that is, when $d \geq 25$.
When $25 \leq d \leq 50$, you'll still have $A(d) = 2d$, but since $2d \geq 50$ you have $A(A(d)) = A(2d) = 3(2d-50)+100$.
Finally, when $d \geq 50$, we have $A(d) = 3(d-50)+100 = 3d-50$; since this is ≥ 50, we have

$$A(A(d)) = A(3d-50) = 3((3d-50)-50)+100 = 3(3d-100)+100.$$

Thus, we have that

$$A(A(d)) = \begin{cases} 4d & 0 \leq d \leq 25 \\ 3(2d-50)+100 & 25 \leq d \leq 50 \\ 3(3d-100)+100 & 50 \leq d. \end{cases}$$

6. [13 points] You're looking at buying two cars:

- Car A is worth \$30,000 initially, and the value decreases by 15% annually.
- Car B is worth \$20,000 initially, and the value also decreases exponentially. Let r be the annual growth rate. Note that r is negative.

a. [3 points] If we know that the values of Car A and Car B will be equal at some point in the future, which of the following must be true? Briefly explain your reasoning.

i. $r < -0.15$.

ii. $r > -0.15$.

iii. We do not have enough information to decide.

Solution: We must have (ii): $r > -.15$. Because Car A starts at a higher value than Car B, if Car B's value were to decay faster than Car A they would never intersect. Thus Car B's value must decay *slower* than Car A. This means that r is less negative, i.e., $r > -.15$.

b. [3 points] Suppose that r is some value so that the cars do eventually become the same price, and then r increases (so r gets closer to 0) and everything else stays the same. Will the time it takes for the two cars to become equal in value increase or decrease? Briefly explain your reasoning.

Solution: Decrease: As r increases (gets closer to 0), the rate at which Car B is declining in value *slow*. Since Car A starts off more valuable than Car B, the slower Car B declines the less Car A has to drop in value before they're equal in value. So, the point of intersection will happen sooner, and thus the time decreases.

c. [4 points] Let t be the number of years from now when the two cars are equal in value. Find t (in exact form). Your answer may contain r.

Solution: The value of Car A is given by $30000(.85)^t$, and the value of Car B by $20000(1+r)^t$. Setting these equal and isolating t in the exponent, we have:

$$\begin{aligned} 20000(1+r)^t &= 30000(.85)^t \\ \frac{(1+r)^t}{.85^t} &= \frac{30000}{20000} \\ \left(\frac{1+r}{.85}\right)^t &= \frac{3}{2}. \end{aligned}$$

Now we take log of both sides (we could also use ln) and use log rules:

$$\begin{aligned} \log\left(\left(\frac{1+r}{.85}\right)^t\right) &= \log\left(\frac{3}{2}\right) \\ t\log\left(\frac{1+r}{.85}\right) &= \log\left(\frac{3}{2}\right) \\ t &= \frac{\log\left(\frac{3}{2}\right)}{\log\left(\frac{1+r}{.85}\right)}. \end{aligned}$$

You can write your answer equally correctly as

$$t = \frac{\log 3 - \log 2}{\log(1+r) - \log(.85)},$$

or with ln in place of log.

d. [3 points] If the cars will be equal in value in 10 years, find r (in exact form).

Solution: We start the same way as part (c): we set the two cars value equal, but with $t = 10$ plugged in. We then solve for r:

$$\begin{aligned} 20000(1+r)^{10} &= 30000(.85)^{10} \\ \left(\frac{1+r}{.85}\right)^{10} &= \frac{3}{2} \end{aligned}$$

Now, we take tenth roots:

$$\begin{aligned} \frac{1+r}{.85} &= \left(\frac{3}{2}\right)^{1/10} \\ 1+r &= .85\cdot\left(\frac{3}{2}\right)^{1/10} \\ r &= .85\cdot\left(\frac{3}{2}\right)^{1/10} - 1. \end{aligned}$$

(You can also do part (d) by taking your answer for part (c), plugging in $t = 10$, and solving for r.)