Math 105 — Second Midterm — December 1, 2020

1. This exam has 10 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Partial credit will be awarded for correct work.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.
5. You do not need to “simplify” your answers unless asked to do so.
6. You must use the methods learned in this course to solve all problems. Logarithm functions taught in this course include “log” (log base 10) and “ln” (natural log).
7. You may use one pre-written page of notes, on an 8.5”x11” standard sheet of paper, with whatever you want written on both sides.
8. You will not be allowed to use any other resources, including calculators, other notes, or the book.
9. You must write your work and answers on **blank, white, physical paper**.
10. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
11. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	2
2	7
3	12
4	10
5	7
6	10
7	12

Total	60

1. [2 points] **There is work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want written on both sides.
- You are not allowed to use any other resources, including calculators, other notes, or the book.
- You may not use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may not use help from any other individuals (other students, tutors, online help forums, etc.), and may not communicate with any other person other about the exam until **8am on Wednesday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [7 points] The following parts are unrelated.

a. [4 points]
An invertible function, $g(x)$, has domain and range all real numbers; the following table gives some specific values.

x	-2	-1	-0.3	0	1	2	3
$g(x)$	5	3	1	0	-1	-2	-8

Using this table, write down exact values for the following expressions, or write "not enough information" if there is no way to tell.

i. $g^{-1}(3)$.
ii. $2g^{-1}(g(101))$
iii. $(g(2))^{-1}$.
iv. $g(g(1))$.

Solution:

i. $g^{-1}(3) = -1$.
ii. $2g^{-1}(g(101)) = 202$.
iii. $(g(2))^{-1} = -1/2$.
iv. $g(g(1)) = g(-1) = 3$.

b. [3 points] Let $C(t) = e^{t^2+1}$. Find possible functions $A(t)$ and $B(t)$ (with $A(t) \neq t$ and $B(t) \neq t$) such that $A(B(t)) = C(t)$. (Note: there are several possible answers!)

Solution: There are multiple possible correct answers. Here are some possibilities:

- $A(t) = e^t$, $B(t) = t^2 + 1$.
- $A(t) = e^{t+1}$, $B(t) = t^2$.
- $A(t) = et, B(t) = e^{t^2}$.

3. [12 points] Oakley gets exercise every day in one of two ways: either by walking outside, or following instructions from an exercise app.

- $f(d)$ is the amount of time, in minutes, it takes Oakley to walk d miles.
- $f(d)$ is invertible.
- $W(t)$ is Oakley's heart rate, in beats per minutes, t minutes after they start walking.
- $A(t)$ is Oakley's heart rate, in beats per minutes, t minutes after they start using the exercise app.

a. [8 points] For each of the following, give a practical interpretation of the given expression, or explain why the expression does not make sense in the context of the problem.

i. $f^{-1}(5)$

ii. $W(f(1.5)) = 95$

iii. $W(40) < A(20)$

Solution:

i. $f^{-1}(5)$ is the distance in miles that Oakley can walk in 5 minutes.

ii. Oakley's heart rate after walking 1.5 miles is 95 beats per minute.

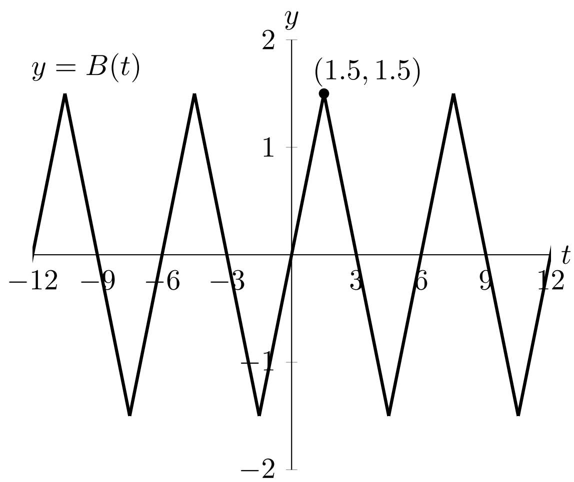
iii. Oakley's heart rate after walking for 40 minutes is less than their heart rate after using the exercise app for 20 minutes.

b. [4 points] Find an expression for Oakley's average speed, in miles per **hour**, when Oakley has walked a total of d miles. Your answer may involve f, W, and/or A.

Solution: To walk d miles takes $f(d)$ minutes, or $f(d)/60$ hours. Since average speed is distance traveled over time, the average speed is

$$\frac{d}{f(d)/60} = \frac{60d}{f(d)}.$$

4. [10 points] The plot below shows a graph of $y = B(t)$, the height in feet of a buoy floating in the ocean t minutes after **6 am.**



Use the graph to answer the following questions:

a. [2 points] What is the period of $B(t)$? Include units.

Solution: The period of $B(t)$ is 6 minutes.

b. [3 points] For each of the following transformations, write down if the the function is even, odd, or neither.

i. $B(t - 7.5) + 1$.

ii. $-B(t) + 2.25$.

iii. $B(-t)$.

Solution:

i. Even.

ii. Neither.

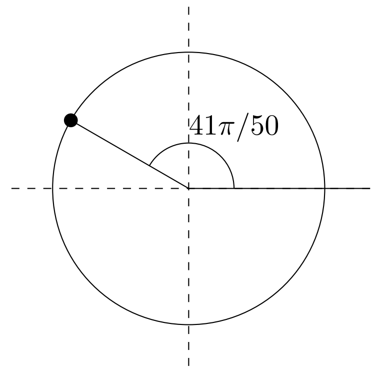
iii. Odd.

c. [5 points] Let $G(h)$ be the function telling you the height in **inches**, at time h **hours** after **8 am**. Write a formula for $G(h)$ in terms of B. (Recall that there are 12 inches in one foot.)

Solution: $G(h) = 12B(60(h - 2))$; this can also be written $12B(60h - 120)$.

5. [7 points] The parts of this problem are unrelated.

a. [2 points] Give another angle θ **in radians**, with $0 \leq \theta \leq 2\pi$, with the same value for cosine as the angle shown below:



Solution: Another angle giving the same cosine value is

$$2\pi - 41\pi/50 = 59\pi/50$$

b. [5 points] The graph of a function $y = M(x)$ has the following properties:

- The amplitude is 4
- The midline is $y = 2$
- The period is 3.
- $y = M(x)$ has a minimum at $x = 0$.

Consider the function $V(x) = 2M(-4x) - 1$. Find the following. For any that cannot be determined from the given information, write "cannot be determined".

i. The amplitude of $y = V(x)$.

ii. The midline of $y = V(x)$.

iii. The period of $y = V(x)$.

iv. The y-intercept of $y = V(x)$.

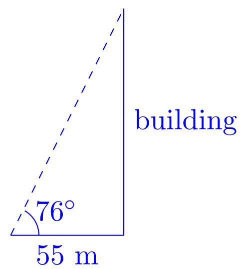
Solution:

- The amplitude is 8.
- The midline is $y = 3$.
- The period is $3/4$.
- The y-intercept is -5.

6. [10 points] In this problem you may assume your height is 0 meters (because you are much, much shorter than a building). You're standing at the base of Michigan's Tallest Building, and you want to know exactly how high it is. Measuring your steps, you walk 55 meters away, look at the top of the building, and measure that your line of sight makes an angle of 76 degrees with the ground.

a. [3 points] Draw a picture of the situation described above. Label all given distances and angles.

Solution:



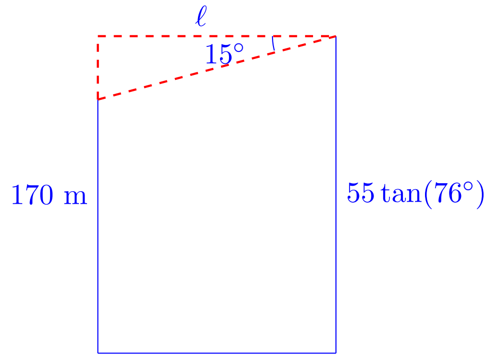
b. [3 points] What is the height of Michigan's Tallest Building? Leave your answer in exact form.

Solution: Call the height h. From the diagram in part (a), we see that:

$$\tan(76^\circ) = \frac{h}{55}$$
$$h = 55\tan(76^\circ).$$

c. [4 points] You go to the top of Michigan's Tallest Building, and look down at a shorter building, which you know to be 170 meters. You want to know how far apart (horizontally) the shorter building and Michigan's Tallest Building are. You observe that the angle between horizontal and your line of sight to the top of the shorter building is 15 degrees. (Note that since you're above the shorter building, you're looking *below* horizontal!) How far away is the shorter building? Your answer may involve your answers to part b. Leave your answer in exact form. (Hint: draw a picture!)

Solution: A diagram of this situations might look like:



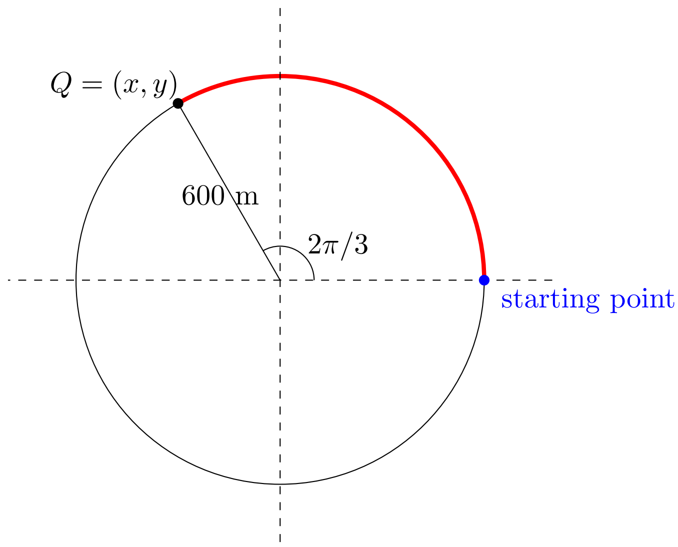
Call the unknown distance we are looking for ℓ. The difference between the two buildings' heights, which is the length of the vertical side of the right triangle with dashed red sides, is

$$55\tan(76^\circ) - 170.$$

Using this triangle, we have that

$$\begin{aligned} \tan(15^\circ) &= \frac{55\tan(76^\circ) - 170}{\ell} \\ \ell &= \frac{55\tan(76^\circ) - 170}{\tan(15^\circ)} \end{aligned}$$

7. [12 points] A race car is traveling around a circular racetrack at **constant** speed. It starts at the 3 o'clock position and moves counter-clockwise around a circular track that has radius 600 meters.
It takes 2/3 of a minute for the car to go from the starting point to the point marked Q.



a. [3 points] What is the speed of the car in meters per minute? Note that the given angle is $2\pi/3$ *radians*.

Solution: We first need the distance the car has traveled. The car has gone 1/3 of the full circumference of the circle. The circumference is $2\pi R$, where $R = 500, 600$, or 700, and so the distance is $2\pi R/3$. Since it took 2/3 of a minute to do so, the speed is

$$\frac{2\pi R/3}{2/3} = \pi R.$$

Depending on your R, this will be 500π, 600π, or 700π.

b. [4 points] Write a formula for $P(t)$, the x-coordinate of the car's position t minutes after the car leaves the start line, where the center of the track is at the origin. Your answer will be a sinusoidal function, and all constants should be left in exact form.

Solution: The formula will be

$$P(t) = R\cos(\pi t)$$

(where again $R = 500, 600$ or 700).

c. [5 points] Using your answer to part **b.**, what are the first two positive values of t (in exact form) at which the x-coordinate of the car is equal to 100?

Solution: To find the first solution, set $P(t) = 100$ and solve using inverse trig functions:

$$\begin{aligned} R\cos(\pi t) &= 100 \\ \cos(\pi t) &= 100/R \qquad (100/R \text{ will be } 1/5,\ 1/6, \text{ or } 1/7) \\ \pi t &= \arccos(100/R) \\ t &= \frac{\arccos(100/R)}{\pi} \end{aligned}$$

To find the second solution, add a period (2 minutes) to the negative of the first solution:

$$t = -\frac{\arccos(100/R)}{\pi} + 2$$

(Again, $100/R$ will be $1/5$, $1/6$, or $1/7$.)