Math 105 — Second Midterm — December 7, 2021

Write your 8-digit UMID number very clearly in the box to the right, and fill out the information on the lines below.

Your Initials Only: ______ Your 8-digit UMID number (not uniqname): ____________

Instructor Name: Solutions Section #: ________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 7 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one calculator that does not have an internet or data connection (scientific or graphing recommended).
10. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is <u>not</u>.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	8	
2	11	
3	11	

Problem	Points	Score
4	5	
5	12	
6	13	
Total	60	

1. [8 points] Parts **a.** and **b.** are unrelated.

a. [4 points] Let $f(x)$ and $h(x)$ be functions given by the formulas:

$$f(x) = \sqrt{1 + \pi\sqrt{x}} \qquad \text{and} \qquad h(x) = \sqrt{x}.$$

You do not need to show work for this part.

(i) Find a formula for a function $s(x)$ such that $f(x) = s(h(x))$.

Answer: $s(x) =$ $\sqrt{1 + \pi x}$

(ii) Find a formula for a function $r(x)$ such that $f(x) = h(r(x))$.

Answer: $r(x) =$ $1 + \pi\sqrt{x}$

b. [4 points] Given the function $K = g(c)$ below, find a formula for $g^{-1}(K)$. *Show all of your work.*

$$K = g(c) = \frac{\ln(c^{10}) - \ln(c^7)}{\log(10^4)}$$

$$K = \frac{10 \cdot \ln(c) - 7\ln(c)}{4}$$

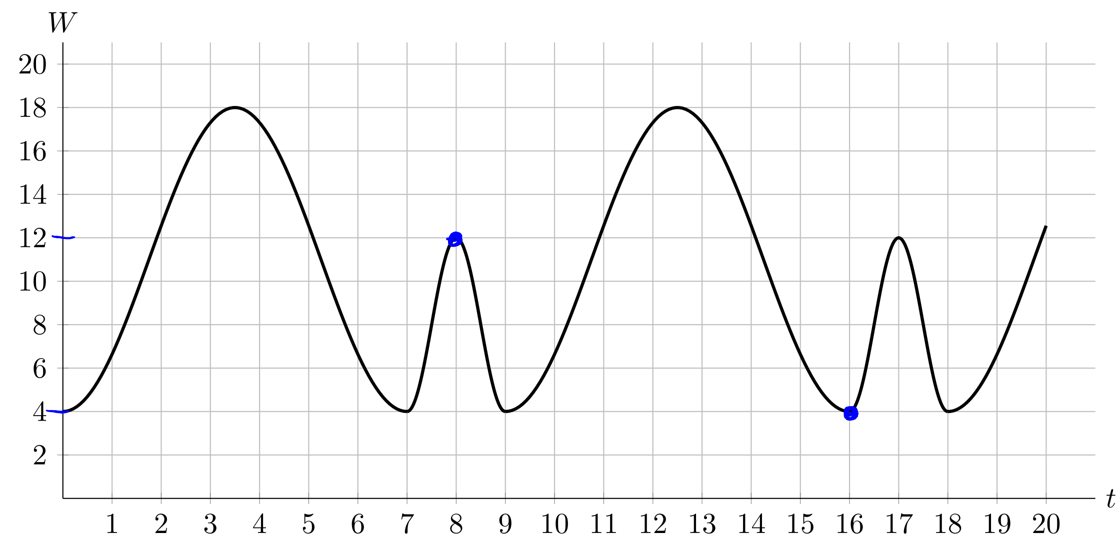
$$4K = 3\ln(c)$$

$$\ln(c) = \frac{4}{3}K$$

$$c = e^{4/3 K}$$

Answer: $g^{-1}(K) =$ $e^{\frac{4}{3}K}$

2. [11 points] As the Smashing Squash are touring, their merchandise varies in value in a way that can be modeled by a periodic function. Let $W = P(t)$ be the value (in thousands of dollars) of an autographed vinyl record at time t (in months). Suppose that $P(t)$ is a periodic function with period less than 18 months. Part of the graph of $W = P(t)$ is shown below.



You do not need to show work for this problem.

a. [1 point] Find the average rate of change of $P(t)$ between $t = 8$ and $t = 16$.

Answer: $\frac{4-12}{16-8} = \frac{-8}{8} = -1.$

b. [2 points] Find the period of $P(t)$. **Include units** in your answer.

Answer: $17-8 = 9$ months

c. [2 points] Find the amplitude of the function $P(t)$. **Include units** in your answer.

Answer: $\frac{18-4}{2} = 7$ thousand dollars

d. [2 points] Find the equation of the midline of the function $P(t)$.

Answer: $W = 4+7 = 11$

e. [2 points] Find the smallest value of t that satisfies $t > 20$ and at which point the record has a value of $4,000.

Answer: $16+9 = 25$

f. [2 points] Let $k(t) = -100P(2t)$. What is the period of $k(t)$?

horizontal compression by a factor of $\frac{1}{2}$

Answer: $\frac{1}{2} \cdot 9 = 4.5$

3. [11 points] Some values of a function $y = f(x)$ are given below.

x	-4	0	1
$f(x)$	0	2	-3

Let $j(x)$ be the function whose graph is obtained by performing the following transformations to the graph of $f(x)$, in the following order:

1. A horizontal compression by a factor of $\frac{1}{2}$. $-2 \quad 0 \quad \frac{1}{2}$
2. A horizontal shift to the right by 1. $-1 \quad 1 \quad \frac{3}{2}$
3. A vertical stretch by a factor of 10. $0 \quad 20 \quad -30$
4. A reflection across the x-axis. $0 \quad -20 \quad 30$

a. [4 points] Use this information to fill out the table below. You do not need to show your work, but partial credit may be awarded for correct work.

x	-1	1	$\frac{3}{2}$
$j(x)$	0	-20	30

b. [4 points] Find a formula for $j(x)$ in terms of $f(x)$.

Answer: $j(x) = -10f\big(2(x-1)\big)$

c. [3 points] Show that your answers to the table and formula match one another.
Note: you can still receive full credit for this part even if you find that they don't match but are not sure how to fix your answers (please note that this is what you have found), or you weren't able to do one or both parts above, but can explain how you would check that the answers match if you did have answers.

Use x-values from table in (a) in formula from (b)

$$-10f\big(2(-1-1)\big) = -10f(-4) = -10 \cdot 0 = 0$$

use table of values for f (not j)

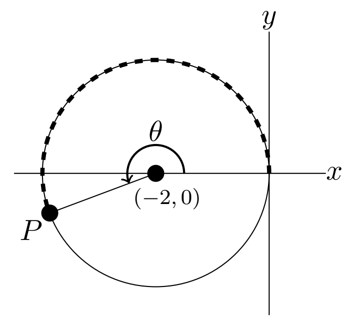
result should be values in table from (a).

$$-10f\big(2(1-1)\big) = -10f(0) = -10 \cdot 2 = -20$$

$$-10f\left(2\left(\tfrac{3}{2}-1\right)\right) = -10f\left(2 \cdot \tfrac{1}{2}\right) = -10 \cdot f(1) = -10 \cdot -3 = 30$$

4. [5 points]

In the diagram at right, a circle of radius 2 is centered at the point $(-2, 0)$. The **bold**, dashed arc going from the origin to the point P has length 7.



(i) Find the exact value of the measure of the angle θ, in radians.

Answer: $\theta =$ $\frac{7}{2}$

(ii) Find the (x, y)-coordinates of the point P.

Answer: $(x, y) =$ $\left(2\cos\left(\frac{7}{2}\right) - 2,\ 2\sin\left(\frac{7}{2}\right)\right)$

5. [12 points] Billy Corgi (the lead singer of The Squash) left his pool uncovered when he went on tour. Due to the rainy weather while he was on tour, the volume, in m^3, of the water in the pool w weeks after he goes on tour is given by $p(w) = 10e^{0.05w}$.

a. [3 points] Find $p^{-1}(50)$ and interpret your answer in the context of this problem. Show your work. Your answer should be in exact form or correct to two decimal places.

$10e^{0.05w} = 50$

$e^{0.05w} = 5$

$0.05w = \ln(5)$

$w = \frac{\ln(5)}{0.05}$

Answer: $p^{-1}(50) =$ $\frac{\ln(5)}{0.05} \approx 32.19$

Interpretation:

The volume of the pool is 50 m^3 when Billy Corgi has been on tour for $\frac{\ln(5)}{0.05}$ weeks.

b. [2 points] What kind of function is the composition $h(w) = \log(p(w))$?

$\log\left(10e^{0.05w}\right)$

$= \log(10) + 0.05w \cdot \log(e)$

i. $h(w)$ is linear

ii. $h(w)$ is quadratic

iii. $h(w)$ is exponential

iv. NONE OF THESE

5., continued. *Restated from the previous page.* Billy Corgi (the lead singer of The Squash) left his pool uncovered when he went on tour. Due to the rainy weather while he was on tour, the volume, in m^3, of the water in the pool w weeks after he goes on tour is given by $p(w) = 10e^{0.05w}$.

Recall from the first exam that the volume (in m^3) of Alex Artakis' pool w weeks after the Neverclear tour began is given by $v(w) = 120(0.9775)^w$. Assume that the bands started their tours at the same time.

c. [4 points] is there a time at which Billy's pool has the same volume of water as Alex's? Either find this value, in exact form or correct to two decimal places, showing step-by-step work, or explain why no such time exists.

$$10e^{0.05w} = 120(0.9775)^w$$

$$e^{0.05w} = 12(0.9775)^w$$

ln of both sides:

$$0.05w = \ln(12(0.9775)^w)$$

$$0.05w = \ln(12) + \ln(0.9775^w)$$

$$0.05w = \ln(12) + w \cdot \ln(0.9775)$$

Subtract $w \cdot \ln(0.9775)$ from both sides:

$$0.05w - w \cdot \ln(0.9775) = \ln(12)$$

$$w(0.05 - \ln(0.9775)) = \ln(12)$$

$$w = \frac{\ln(12)}{0.05 - \ln(0.9775)} \text{ weeks}$$

$$\approx 34.15$$

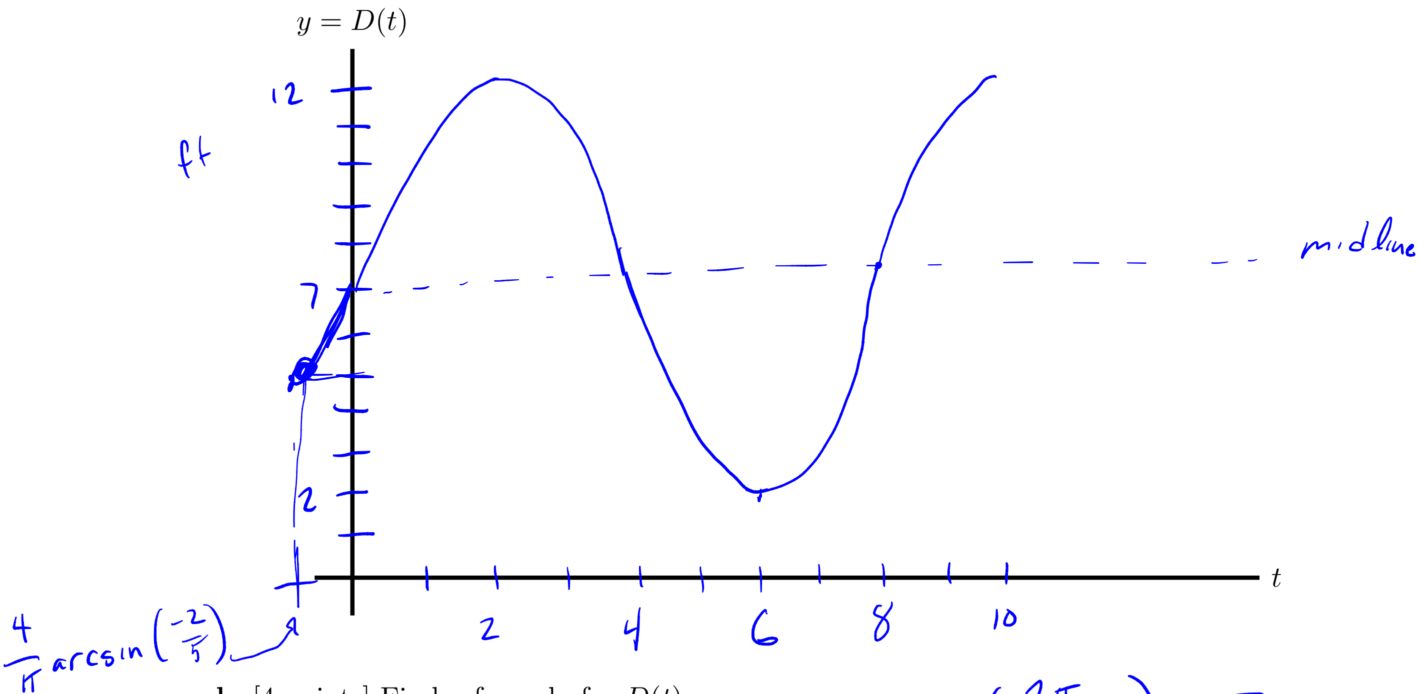
d. [3 points] Give a practical interpretation of the expression $v^{-1}(p(8))$, or explain why the expression doesn't make sense in the context of the problem. You do **not** need to evaluate this expression.

$v^{-1}(p(8))$ weeks into the tours, Alex's pool will contain the same amount of water as Billy's pool did 8 weeks into the tour.

6. [13 points] The alternative rock band π'd-Eye-Blind was unhappy with the first music video and are trying again. There is just one camera on a drone, which again moves in a circle at a constant speed, between a maximum of 12 feet above the ground and a minimum of 2 feet above the ground. At time $t = 0$ it is halfway between the maximum and minimum height, and moving upward. It makes one full circle in 8 seconds.

Let $D(t)$ be the height of the drone t seconds after they start filming.

a. [4 points] Sketch a graph of $y = D(t)$ for $0 \le t \le 10$. Be sure to label your axes, and pay careful attention to the shape of your graph.



b. [4 points] Find a formula for $D(t)$.

Answer: $D(t) =$ $5\sin\left(\frac{2\pi}{8}t\right) + 7$

c. [5 points] Find all times at which the drone is exactly 5 feet above the ground for $0 \le t \le 10$. Show your work, and give your answers in exact form.

Note: based on graph, there should be $\underline{2}$ such times.

$$5\sin\left(\frac{\pi}{4}t\right) + 7 = 5$$

$$5\sin\left(\frac{\pi}{4}t\right) = -2$$

$$\sin\left(\frac{\pi}{4}t\right) = -\frac{2}{5}$$

One time:

$$\frac{\pi}{4}t = \arcsin\left(\frac{-2}{5}\right)$$

$$t = \frac{4}{\pi}\arcsin\left(\frac{-2}{5}\right)$$

but this is < 0.

$$\boxed{t = 8 + \frac{4}{\pi}\arcsin\left(\frac{-2}{5}\right) \text{ and } 4 - \frac{4}{\pi}\arcsin\left(\frac{-2}{5}\right).}$$