Math 105 — First Midterm — October 6, 2022

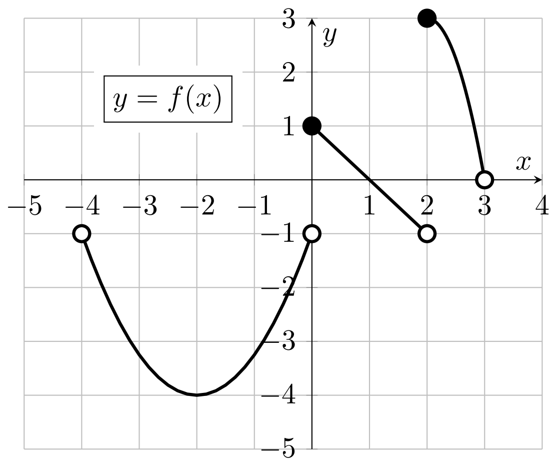
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 8 pages including this cover. There are 7 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one calculator that does not have an internet or data connection (scientific or graphing recommended).
10. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	10	
3	10	
4	10	

Problem	Points	Score
5	9	
6	8	
7	4	
Total	60	

1. [9 points] The entire graph of a function $f(x)$ is shown below to the left. Also shown is a table of some values for the functions $p(x)$ and $r(x)$. Assume that the function $p(x)$ is invertible.



x	-1	0	1	2	5
$p(x)$	4	2	-1	-3	-5
$r(x)$	3	1	-4	-2	0

a. [3 points] Find the domain and range of $f(x)$. Give your answers using interval notation or using inequalities. *You do not need to explain or justify your answer.*

Answer: Domain: $(-4, 3)$

Range: $[-4, -1) \cup (-1, 3]$

b. [2 points] Calculate the average rate of change of $f(x)$ on the interval $-2 \le x \le 2$. *Partial credit may be awarded for work shown.*

Answer: $\frac{3 - -4}{2 - -2} = \frac{7}{4}$

c. [4 points] Find each of the following, or write N/A if a value does not exist or there is not enough information to find it.
You do not need to show work.

(i) $p^{-1}(2)$ **Answer:** $p^{-1}(2) =$ 0

(ii) $f(p(0))$ **Answer:** $f(p(0)) =$ 3

(iii) $f(r(1))$ **Answer:** $f(r(1)) =$ **N/A**

(iv) If $g(x) = f(x - 2) + 3$, find $g(0)$. **Answer:** $g(0) =$ -1

2. [10 points] Jada has a new job, selling candy to the elves.
Below is a chart that shows how much money, D, in dollars, Jada has in her cash register h hours after she starts selling candy.

h	3	5	6
D	16	25	?

a. [2 points] What assumption needs to be made about the situation in order for it to be reasonable to model D using a **linear** function of h?

Solution: We need to assume that the amount of money in Jada's cash register changes at a constant rate per hour—that is, that she takes in the same amount each hour.

b. [1 point] If D can be modeled as a linear function of h, how much money, in dollars, will Jada have after 6 hours?

Answer: **29.5**

c. [4 points] Find both the slope and vertical intercept of the linear function. Then, for each quantity, write a sentence interpreting that quantity in the context of the problem.

Answer: Slope: **4.5**

Interpretation:

Solution: The amount of money in Jada's cash register increases by \$4.5 each hour.

Answer: Vertical intercept: **2.5**

Interpretation:

Solution: Jada starts with \$2.50 in her cash register.

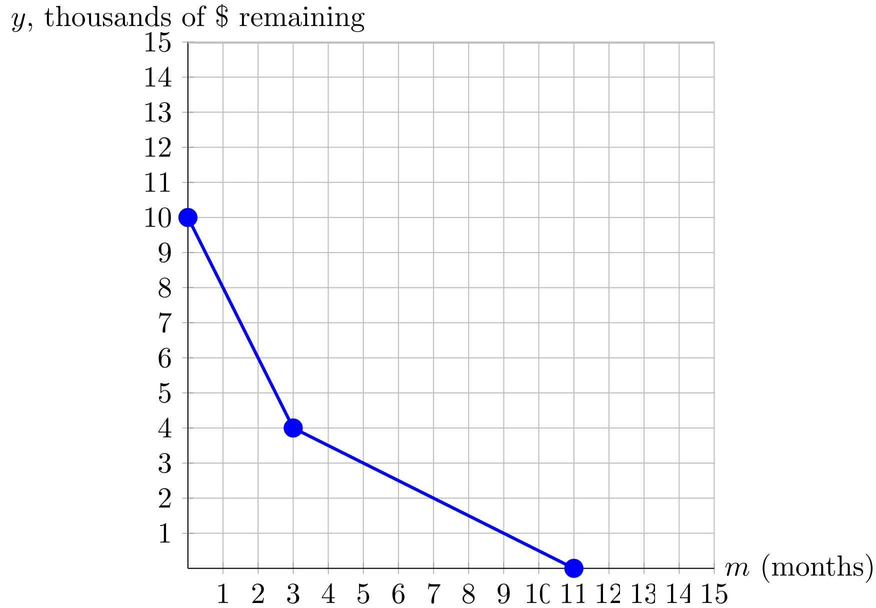
d. [3 points] Jada is selling both chocolate bars for \$0.20 each and lollipops for \$0.05 each. Suppose that Jada makes exactly \$8 one day selling B chocolate bars and P lollipops. Let f be the function such that $B = f(P)$ in this case. Find a formula for f.

Solution: Based on the information given, we can set up the equation $0.2B + 0.05P = 8$. Then, in order to find B as a function of P, we need to solve the equation for B, giving us a formula in terms of P. We do this by subtracting $0.05P$ from both sides, and then dividing both sides by 0.2.

Answer: $f(P) = \dfrac{8 - 0.05P}{0.2} = 40 - 0.25P$

3. [10 points] Janice recently won \$10,000 through the Michigan lottery. She was so excited to have the extra spending money that she spent her winnings at a constant rate of \$2,000 per month. However, when she had \$4,000 remaining, she decided to curb her spending to make the money last, and she decreased her spending to a constant rate of \$500 per month until she spent all of her winnings. Let $W(m)$ be the remaining amount of money, in thousands of dollars, that Janice has left from her lottery winnings m months after she wins the money.

a. [4 points] Draw a graph of your function $W(m)$. Be sure to label your axes (including units) along with any important points, including the beginning and end of different pieces of your graph.



b. [1 point] After how many months does she spend all of her winnings?

Solution: 11 months. We can see this from the graph, or from noting that she gets to \$4000 after spending $10,000 - 6,000$, and she will spend \$6,000 after $6000/2000 = 3$ months. She will spend the remaining money in another $4000/500 = 8$ months, meaning it will all be spent after a total of $3 + 8 = 11$ months.

c. [5 points] Find a piecewise-defined formula for $W(m)$ on the appropriate domain in the context of the problem.

Solution: For the first piece, we know that she starts with 10 thousand dollars and spends 2 thousand a month, giving us $10-2m$, and we found previously that this was on the interval $0 \leq m \leq 3$. After that, we know that the slope is -0.5 (thousands of dollars per month), and we know that the second part passes through the point $(3, 4)$, so we can use point-slope form to find a formula. In the previous parts, we found the domain of this piece to be $3 < m \leq 11$. (Note that we could have included the 3 in either domain, since they both give 4 as the corresponding output.)

$$W(m) = \begin{cases} 10 - 2m & \text{if } 0 \leq m \leq 3 \\ -0.5(m-3) + 4 & \text{if } 3 < m \leq 11 \end{cases}$$

4. [10 points] Rita adopts a puppy, which she names Spot, and she records certain information about Spot for the first month she has him.

- Let $f(t)$ be the amount of food, in cups, that Spot eats t days after Rita adopts Spot.
- Let $w(t)$ be Spot's weight, in pounds, t days after Rita adopts Spot.
  - Assume that w is invertible during this month.

a. [8 points] For each of the following, either give a practical interpretation of the mathematical expression or equation, or explain why it doesn't make sense in the context of the problem.

(i) $f(8) = 0.75$

Solution: Spot eats 0.75 cups of food 8 days after Rita gets him.

(ii) $f(w^{-1}(10))$

Solution: $f(w^{-1}(10))$ is the amount of food in cups Spot eats when he weighs 10 pounds.

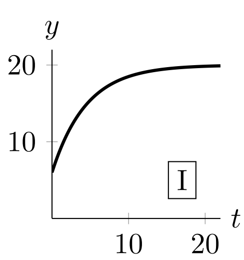
(iii) $w(f(9))$

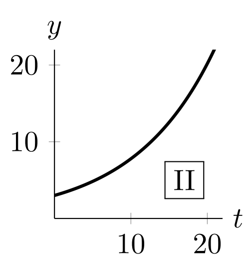
Solution: This expression does not make sense. $f(9)$ gives us an amount of food (in cups), but the input for w should instead be days.

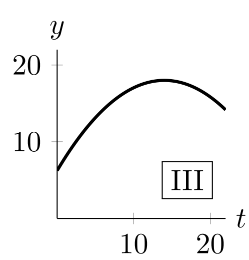
(iv) $\dfrac{f(14) - f(7)}{14 - 7} = 0.05$

Solution: Between 7 and 14 days after Rita gets him, Spot eats an average of 0.05 more cups of food each day than the previous day.

b. [2 points] Spot gains weight every day. When Rita first gets Spot, he is gaining weight very quickly, but he starts growing more slowly later on. Which of the following graphs could be a graph of $y = w(t)$? Choose the **one** best answer.







Answer:

[I] II III

5. [9 points] Danica is tossing her wadded-up notes into a wastebasket across the room. The height above the ground of one particular wad, H (in feet), can be expressed as a function of the horizontal distance, d (in feet), from where Danica releases the wad of paper using the following function:

$$H = -\frac{1}{4}\left(d^2 - 4d - 12\right)$$

a. [2 points] Find the value of the vertical intercept and interpret its meaning in the context of the problem.

Answer: Vertical intercept: 3

Interpretation:

Solution: When Danica releases the wad of paper, it is 3 feet from the ground.

b. [3 points] The rim of the wastebasket is 1.5 feet above the ground and 7 feet horizontally from where Danica released the wad of paper. Using this information, can you tell whether Danica succeeds in throwing the wadinto the wastebasket?

Show all calculations and justify your conclusion with one sentence.

Answer *(circle one)*: She succeeds | **[She fails]** | Cannot be determined

Justification:

Solution: There are a few ways to reach this conclusion. One is to use the formula to find that when $d = 7$, $H = -2.25$. The fact that the height is negative tells us that the wad of paper has already hit the floor earlier, and this is no longer a good model for the height of the paper. Another way is to factor the formula to find $H = -\frac{1}{4}(d+2)(d-6)$. This tells us that the paper hits the floor 6 feet away from Danica, a full foot short of the wastebasket.

c. [4 points] What is the highest point above the ground the wad reaches? Include units.

There are at least two methods you could use here: finding the axis of symmetry using the zeros or completing the square.

Solution: This question is asking us to find the value of H at the vertex of the quadratic function. In the previous part, we found that the zeroes are at $d = -2$ and $d = 6$. This means that the vertex is exactly halfway in between, at $d = 2$. Now we can plug this into the formula to find that $H = \frac{1}{4}(2^2 - 4 \cdot 2 - 12) = 4$ feet when $d = 2$.

Answer: **4 feet**

6. [8 points] The parts of this problem are unrelated.

a. [5 points] The following table gives the values of the variables x, A, B, and C:

x	-1	1	2	4
$A = a(x)$	1	-2	-4	-2
$B = b(x)$	4	3	2	-1
$C = c(x)$	1	5	7	11

(i) Given the values in the tables above, which of the following statements **could** be true? Circle all that apply.

[A is a function of B] B is a function of A None of these

(ii) Which of the functions could be (or are) concave down on the **entire** interval $-1 \leq x \leq 4$? Circle all that could be correct, and justify your answers algebraically.

Answer: $a(x)$ [$b(x)$] $c(x)$ none of these

Justification:

Solution: We can decide the potential concavity of these functions by looking at the average rate of change on each pair of intervals given. This gives us the following values:

interval	$[-1, 1]$	$[1, 2]$	$[2, 4]$
$A = a(x)$	$-3/2$	$-2/1 = -2$	$+1$
$B = b(x)$	$-1/2$	-1	$-3/2$
$C = c(x)$	$4/2 = 2$	$2/1$	$4/2 = 2$

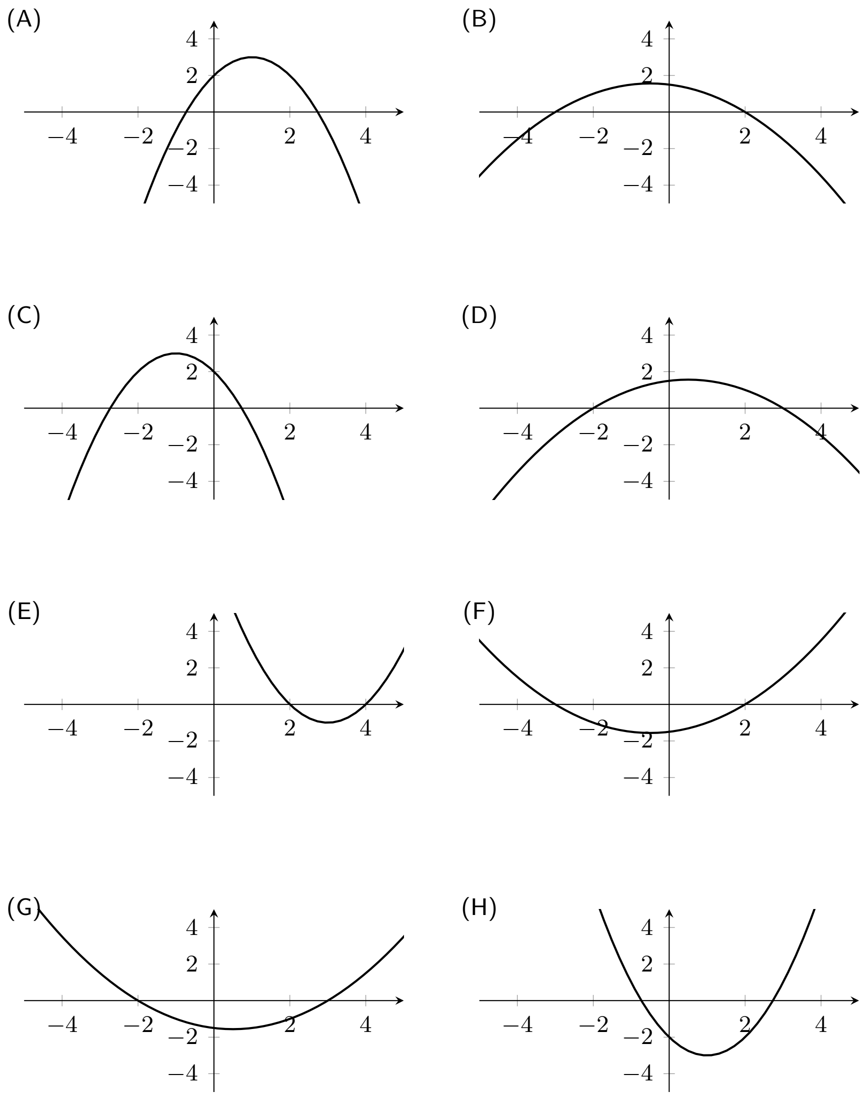
Based on this information, $c(x)$ could be linear, but not concave up or down, since the average rates of change are the same on all intervals. The average rates of change of $a(x)$ decrease from the first interval to the second, but increase from the second interval to the third, indicating that it cannot be concave up or down on the entire interval either. On the other hand, we see $-1/2 > -1 > -3/2$, so $b(x)$ could be concave down on this interval, as its average rates of change are decreasing.

b. [3 points] Two lines are given by the equations $y = Kx + 5$ and $x + y = 4$, where K is some constant. For what value(s) of K, if any, will these two lines intersect at $x = 1$? Show your work or explain your reasoning.

Solution: There are a number of ways to solve this. One is to plug $x = 1$ into the second equation, giving $y = 3$. Putting both these values into the first equation gives $3 = K + 5$, meaning $K = -2$.

Answer: $K =$ -2

7. [4 points] Below there are two formulas of quadratic functions and 8 graphs. Choose the **one** graph that best fits each formula below. You do not need to justify your answers.



a. [2 points] Which graph best represents $p(x) = \frac{1}{4}(x-2)(x+3)$?

Answer: F

b. [2 points] Which graph best represents $q(x) = -(x-1)^2 + 3$?

Answer: A