Math 105 — Second Midterm — November 17, 2022

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 8 pages including this cover. There are 7 problems.
   Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one calculator that does not have an internet or data connection (scientific or graphing recommended).
10. If you use a graph or table to find an answer, be sure to sketch the graph or write out the entries of the table. In either case, include an explanation of how the graph or table gives the answer.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form*. Recall that $x = \sqrt{2}$ is a solution in exact form to the equation $x^2 = 2$, but $x = 1.41421356237$ is not.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	4	
3	11	
4	7	

Problem	Points	Score
5	5	
6	11	
7	13	
Total	60	

1. [9 points] Consider the table of known values for the functions $f(x)$ and $h(x)$, where $f(x)$ is invertible.

x	-4	-2	-1	0	1	2	4
$f(x)$	2	3	0	-2	-1	4	5
$h(x)$	?	2	1	4	0	?	7

a. [4 points] Find each of the following, or write N/A if a value does not exist or there is not enough information to find it.

(i) $f^{-1}(0)$ **Answer:** $f^{-1}(0) =$ -1

(ii) $f(h(0))$ **Answer:** $f(h(0)) =$ **5**

(iii) $h(g(1))$, where $g(x) = \log(x)$
Answer: $h(g(1)) =$ **4**

(iv) $k(1)$, where $k(x) = -4f(2(x+1)) - 6$
Answer: $k(1) =$ -26

b. [2 points] If $f(h(2)) = 0$, then what is $h(2)$?
Answer: $h(2) =$ -1

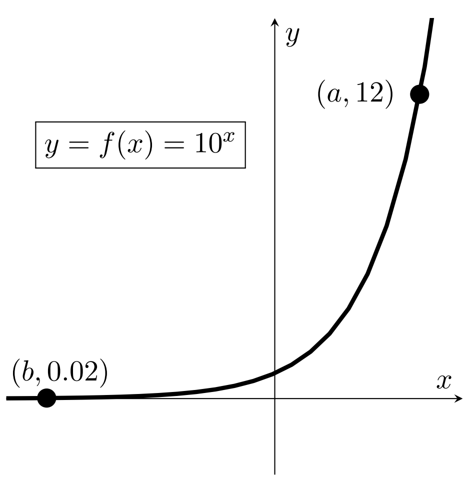
c. [3 points] Give a value for $h(-4)$ that would guarantee that $h(x)$ is *not* invertible and explain (in at most 1 sentence) why your value for $h(-4)$ forces the function to be non-invertible.

Answer: $h(-4) =$ **7 (or any of: 2, 1, 4, 0)**

Explanation:

Solution: If $h(-4) = 7 = h(4)$, then we have two inputs of h with the same output.

2. [4 points] Use the graph of $y = 10^x$ below to decide whether each of the following statements is true (T), false (F), or there is not enough information to tell (NEI).



(i) $a < 1$ T $\boxed{\text{F}}$ NEI

(ii) $b < -1$ $\boxed{\text{T}}$ F NEI

(iii) $\log(12) = a$ $\boxed{\text{T}}$ F NEI

(iv) $\log(b) = 0.02$ T $\boxed{\text{F}}$ NEI

3. [11 points] Three bacteria colonies, called A, B, and C, are established at the same time. The number of bacteria in these colonies are given by $A(t)$, $B(t)$, and $C(t)$, where t is measured in hours since the colonies were established. The formulas for these functions are

$$B(t) = 500 \cdot 3^{t+1}$$
$$C(t) = 100 \cdot e^{2t}$$

a. [1 point] How many bacteria did Colony B start with?

Answer: $500 \cdot 3 = 1500$

b. [1 point] Which, if any, colonies have a percent growth rate of 200% per hour? Circle **all** that are correct.

A $\boxed{\text{B}}$ C None

c. [6 points]

(i) Starting from the time the colonies were established, will colonies A and B ever have the same number of bacteria? If so, find the time when this happens, in exact form or rounded to at least two decimal places. If not, briefly explain why not.

Answer *(circle one):* Yes: $t =$ ________ $\boxed{\text{No}}$ (explain below)

Solution: B starts out with a larger population and grows at a faster rate, so they will never be equal.

(ii) Starting from the time the colonies were established, will colonies A and C ever have the same number of bacteria? If so, find the time when this happens, in exact form or rounded to at least two decimal places. If not, briefly explain why not.

Answer *(circle one):* $\boxed{\text{Yes:}}$ $t = \dfrac{\ln(2)}{2-\ln(2)}$ No (explain below)

Solution: These will be the same when $A(t) = C(t)$, or $200 \cdot 2^t = 100 \cdot e^{2t}$. Solving for t, we find:

$$200 \cdot 2^t = 100 \cdot e^{2t}$$
$$2 \cdot 2^t = e^{2t}$$
$$\ln(2 \cdot 2^t) = \ln(e^{2t})$$
$$\ln(2) + t \cdot \ln(2) = 2t$$
$$2t - t\ln(2) = \ln(2)$$
$$t(2 - \ln(2)) = \ln(2)$$
$$t = \frac{\ln(2)}{2 - \ln(2)} \approx 1.885$$

Recall: A bacteria colony C has population $C(t)$, where t is measured in hours since the colony was established. The formulas for this function is

$$C(t) = 100 \cdot e^{2t}$$

d. [3 points] Find a formula for $g(P)$, a function that gives the amount of time (in hours) it takes for colony C to reach P bacteria.

Solution: We are being asked to find $C^{-1}(P)$. We do this by solving $P = C(t)$ for t:

$$\begin{aligned} P &= 100e^{2t} \\ P/100 &= e^{2t} \\ \ln(P/100) &= 2t \\ t &= \frac{1}{2}\ln(P/100) \end{aligned}$$

Answer: $g(P) =$ $\frac{1}{2}\ln(P/100)$

4. [7 points] Let $g(x) = 2 \cdot (0.5)^{-3x} - 6$.

a. [5 points] List the transformations you need to apply to the graph of $y = 0.5^x$ to transform it to that of $y = g(x)$. Fill each space with either a number or one of the phrases below, as appropriate. (Leave the second blank empty for reflections.)

Solution: One note: the only order here that matters is that the vertical stretch happens before the shift down.

First, **reflect it across the y-axis** by ______

then, **Compress it horizontally** by 1/3

then, **stretch it vertically** by **2**

then, **shift it down** by **6**

b. [2 points] Give equations for all vertical and horizontal asymptotes of $g(x)$. If there are none, write None.

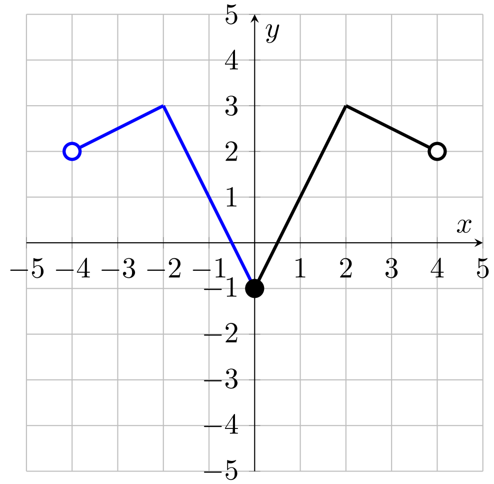
Answer: Vertical Asymptotes: **None**

Answer: Horizontal Asymptotes: $y = -6$

5. [5 points]

a. On the axes below, part of the graph of a function $y = f(x)$ is given. Either draw in the rest of the graph to make the function **even**, or briefly explain why this is not possible.

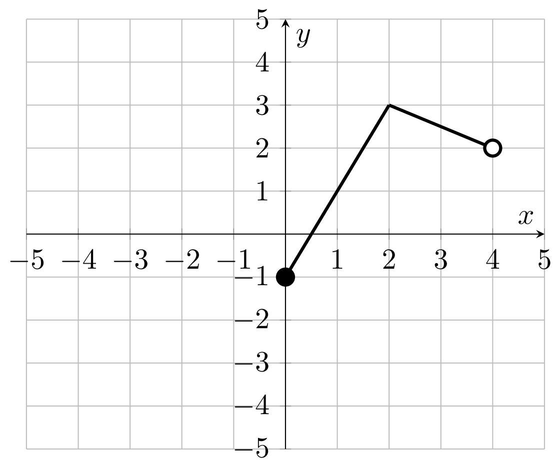
Answer *(Circle one)*: POSSIBLE NOT POSSIBLE



Explanation:

b. On the axes below, part of the graph of a function $y = g(x)$ is given. Either draw in the rest of the graph to make the function **odd**, or briefly explain why this is not possible.

Answer *(Circle one)*: POSSIBLE NOT POSSIBLE



Explanation: Not possible. Note that 0 is in the domain, with $g(0) = -1$. If g were odd, then we would have $g(-0) = -(-1) = 1$, but $g(-0) = g(0)$, so we would need to have two different values for $g(0)$, which isn't allowed for a function.

(Note that it isn't quite true that any odd function must pass through the origin—there is also the possibility that 0 is not in the domain. For example, the function $y = 1/x$ is odd, even though it does not pass through the origin.)

6. [11 points] A video is posted online and later goes viral after it is shared by a certain celebrity on a social media platform. 2 hours after it is shared, it has 5 thousand views, and 6 hours after it is shared, it has 10 thousand views.

a. [2 points] Suppose that the number of views increases at a constant rate of views per hour. Find a formula for $f(t)$, the number of views, in thousands, that the video has t hours after it is shared.

Solution: If the number of views per hour is increasing at a constant rate, then this must be a linear function. We find the slope: $\dfrac{10-5}{6-2} = \dfrac{5}{4}$ and then can either put it in point-slope form or find the vertical intercept of 5/2.

Answer: $f(t) = \underline{\frac{5}{4}t + \frac{5}{2} = 1.25t + 2.5}$

b. [4 points] Suppose instead that the number of views increases at a constant **percent** growth rate, find a formula for $g(t)$, the number of views, in thousands, that the video has t hours after it is shared.

Solution: A constant percent growth rate means that the function is exponential, so it must be of the form $g(t) = ab^t$. We plug in the two points we know: $ab^2 = 5$ and $ab^6 = 10$, and then solve for b and then a.

Answer: $g(t) = \underline{\left(\frac{5}{2^{1/2}}\right)(2^{1/4})^t}$

c. [2 points] Suppose that the number of views increases at a constant percent growth rate and M is a number greater than 4. Which of the following numbers is **greater**?

- Let A be the time, in hours, it takes for the number of views to increase from 4 thousand to 12 thousand.
- Let B be the time, in hours, it takes for the number of views to increase from M thousand to $3M$ thousand.

Answer *(Circle one)*:

A is greater B is greater $\boxed{\text{They are equal}}$ Cannot be determined

Solution: We are being asked here about the *tripling time* of an exponential function. By its nature, growing by a constant percent growth rate, it will always take the same amount of time to grow by the same percentage, regardless of what the starting amount is. So the amount of time it takes for an exponential function to triple from 4 to 12 is the same as the amount of time it takes for the same function to triple from M to $3M$.

d. [3 points] Another video has gone viral, and the number of views for that video increases by 40% in 2 hours. Find the **continuous** hourly percent growth rate of the number of views of this video. Give your answer in exact form or correct to at least two decimal places.

We know $ae^{2k} = 1.4a$, and want to find k as a percent. Dividing both sides by a and then taking ln of both sides, we get $2k = \ln(1.4)$, so $k = \ln(1.4)/2$. To find this as a percentage, we need to multiply by 100.

Answer: Continuous hourly percent growth rate: $\underline{(\ln(1.4)/2)\cdot 100 \approx 16.82}$ %

7. [13 points] Jada is riding on a train to visit Emerald Beach. Her friend Eva is also traveling to Emerald Beach, but she left on an earlier train. Eva has been tracking her remaining distance to Emerald Beach in sprites, which is the elves' measurement for distance.

- The function $E(m)$ gives Eva's distance away from Emerald Beach, in sprites, m minutes after Eva's train departed.
- Jada's train left 30 minutes after Eva's train.
- When Jada's train departs, it is 25 sprites behind Eva's train.
- At any given moment, both trains are traveling at the same speed, so Jada's train is always exactly 25 sprites farther away from Emerald Beach than Eva's train.

You do not need to show work for this problem.

a. [2 points] Let $J(t)$ be Jada's distance from Emerald Beach, in sprites, t minutes after *Jada's* train departed. Find a formula for $J(t)$ in terms of the function E and the variable t.

Answer: $J(t) =$ $E(t+30)+25$

b. [2 points] Give a practical interpretation of the following equation:

$$E(90) = \frac{1}{2}J(0)$$

Interpretation:

Solution: Eva's distance from Emerald Beach 90 minutes after she left is half Jada's distance from the beach when she left.

c. [5 points] Jada prefers to use earthly measurements of distance, and she knows that one sprite is exactly three miles.
Based on the given information, fill in the following blanks:

When Jada has been traveling for 2 **hours**, Eva has been traveling for 150 minutes.

When Eva is 15 sprites from Emerald Beach, Jada is 120 miles from Emerald Beach.

Let $D(h)$ be Jada's distance from Emerald Beach, in **miles**, h **hours** after *Jada's train* departed. Find a formula for $D(h)$ in terms of the function E and the variable h.

Answer: $D(h) =$ $3E(60(h+0.5))+75 = 3(E(60h+30)+25)$

d. [4 points] Some points of the function $E(m)$ are given in the table below. Using these points, find the coordinates of points that must be on the graph of $D(h)$.

m	30	105	150
$E(m)$	185	80	15

h	0	1.25	2
$D(h)$	630	315	120