Math 105 — First Midterm — October 10, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. Use a pencil for "bubble-in" questions so that you can easily erase your answer if you change your mind.
4. This exam has 11 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
6. The back of every page of the exam is blank, and, if needed, you may use this space for scratchwork. Clearly identify any of this work that you would like to have graded.
7. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
8. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
9. You must use the methods learned in this course to solve all problems.
10. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and $x = 0.424$ would be the same answer expressed in decimal form.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	10	
2	11	
3	12	

Problem	Points	Score
4	7	
5	11	
6	9	
Total	60	

1. [10 points] You are looking to model the growth of a new TikTok hashtag `#Math105FUN` and you have some data to help you. Initially, at time $t = 0$, there are 100 videos with this hashtag. Ten days later (at time $t = 10$), there are 500 videos with this hashtag.

a. [3 points] If you assume the growth of this hashtag is linear, find an expression for the function $L(t)$ giving the number of videos with the hashtag `#Math105FUN` as a function of t given in days. Your function should match the data points you have so far.

$L(t) =$ $100 + 40t$

b. [3 points] If you assume, instead, that the growth of this hashtag is exponential, find an expression for the function $E(t)$ giving the number of videos with the hashtag `#Math105FUN` as a function of t given in days. Your function should match the data points you have so far.

Solution: We are told the function is exponential, and we know its initial value is 100. We need to find its growth factor. Since it grows by a factor of 5 in 10 hours, it will grow by a factor of $5^{\frac{1}{10}}$ each hour. Putting that together we get the formula below.

$E(t) =$ $100 \cdot (5^{\frac{1}{10}})^t$

c. [2 points] You later get another piece of data: at day $t = 12$, the number of videos with the hashtag is 690. Which model—$L(t)$ vs. $E(t)$—better fits this new information? *Show all work.*

Solution: We can plug $t = 12$ into both our models to see which output is closer to 690.

$$L(12) = 100 + 40(12) = 580$$

$$E(12) = 100 \cdot (5^{\frac{1}{10}})^{12} = 689.86$$

From this we see that this new data means that $E(t)$ is a better fit.

(Circle one) $L(t)$ IS A BETTER FIT $\boxed{E(t) \text{ IS A BETTER FIT}}$

d. [2 points] Let $H(t)$ denote the total number of videos with a different hashtag — `#Math105studyfest` — t days after September 20, 2023. We want a new function $G(s)$ that instead denotes the total number of `#Math105studyfest` videos s days after September 30, 2023. How can we write $G(s)$ in terms of $H(t)$?

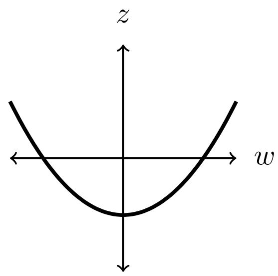
$G(s) = \ldots$ (Circle the best answer)

Solution:
One way to see this is to notice that when we compute $G(0)$ we should get the number of videos with the hashtag on September 30, so $H(10)$. This concrete point helps us to see that we want $G(s) = H(s+10)$.

$H(s-10)$ $\boxed{H(s+10)}$ $H(s)+10$ $H(s)-10$

2. [11 points]

a. [5 points] Each of the following describes a relationship between variables w and z. Fill in the bubble completely for each case where (from the information given) z could be a function of w.



$z - 5w = 3w + 2$

z is the number of people in the M-36 Cafe at w minutes past opening on January 1, 2023.

z is the number of minutes past opening on January 1, 2023 when there are w people in the M-36 cafe.

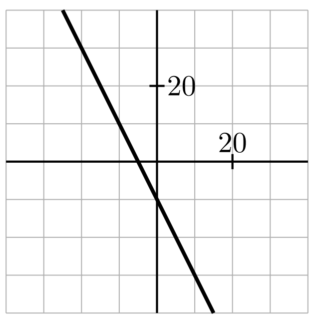
w	0	2	2
z	3	6	1

b. [6 points] Below are several different situations where the variable y can be considered a linear function of x. For each function described, what is the slope of its graph?

i. $y = 2x + 2(x - 1) + 6$ SLOPE = **4**

ii. $y = 5$ SLOPE = **0**

iii. The line going through the points $(-1, -5)$ and $(2, 4)$ SLOPE = **3**

iv. 

SLOPE = -2

v. A line $y = f(x)$ perpendicular to the graph of the line $g(x) = \frac{1}{3}x - 5$ SLOPE = -3

vi. The slope of the line which is the *shift* of the graph of $y = 0.4x - 1$ up by 2 units and left by 5 units. SLOPE = 0.4

3. [12 points] A heater is turned on in a cold room. Let $n = f(T)$ be the number of hours it takes for the heater to warm the room to a temperature of T degrees Fahrenheit (°F). A table of values of this function is given below.

T	61	64	66	67	68
$n = f(T)$	0.5	1.3	2.3	3.3	7

The cost, C, in dollars, to run the heater for n hours is given by the formula

$$C = g(n) = 0.25 + 0.4n.$$

Both f and g are invertible functions.

a. [2 points] Compute the quantities $f^{-1}(0.5)$ and $g(f(68))$.

Solution: $g(f(68)) = g(7) = 0.25 + 0.4(7) = 0.25 + 2.8 = 3.05$

Answer: $f^{-1}(0.5) =$ **61** and $g(f(68)) =$ **3.05**

b. [2 points] Find a formula for g^{-1} in terms of C.

Solution: To get the inverse we can solve $C = 0.25 + 0.4n$ for n as follows:

$$C - 0.25 = 0.4n$$

$$\frac{C - 0.25}{0.4} = n$$

This gives us a formula for n in terms of C, which is our inverse.

Answer: $g^{-1}(C) = \frac{C-0.25}{0.4}$

c. [3 points] For each part below, write a phrase or sentence giving a practical interpretation of the given expression or equation, or explain why it doesn't make sense in this context.

i. $g(1) = 0.65$

Solution: The cost to run the heater for 1 hour is \$0.65.

ii. $f(g(3))$

Solution: This composition does not having a meaning in this context. The units of $g(3)$ are dollars, which does not make sense to plug into f, which takes a temperature in degrees Fahrenheit.

(Problem continues on the next page.)

(The problem has been restated here for convenience.)

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Both f and g are invertible functions.

d. [3 points] For each item below, write an expression or equation, possibly involving the functions f, g, and/or their inverses, that represents the given statement.

i. It takes an hour to heat the room to 63 °F.

Solution: $f(63) = 1$ OR $f^{-1}(1) = 63$.

ii. the temperature of the room when the heating costs have reached $1

Solution: $f^{-1}(g^{-1}(1))$. It is also fine if you solve for $g^{-1}(1)$ explicitly and get instead: $f^{-1}(0.75/0.4)$ or $f^{-1}(1.875)$

e. [2 points] Circle the numeral of the one description below that is best supported by the evidence in this problem. Clearly show your work in the space below.

i.Each °F increase in temperature takes the same amount of time.

$\boxed{\text{ii.}}$ As the room warms up, it takes an increasing amount of time to heat the room to each additional °F in temperature.

iii.It takes less and less time for the heater to heat the room to each additional °F in temperature.

Work:

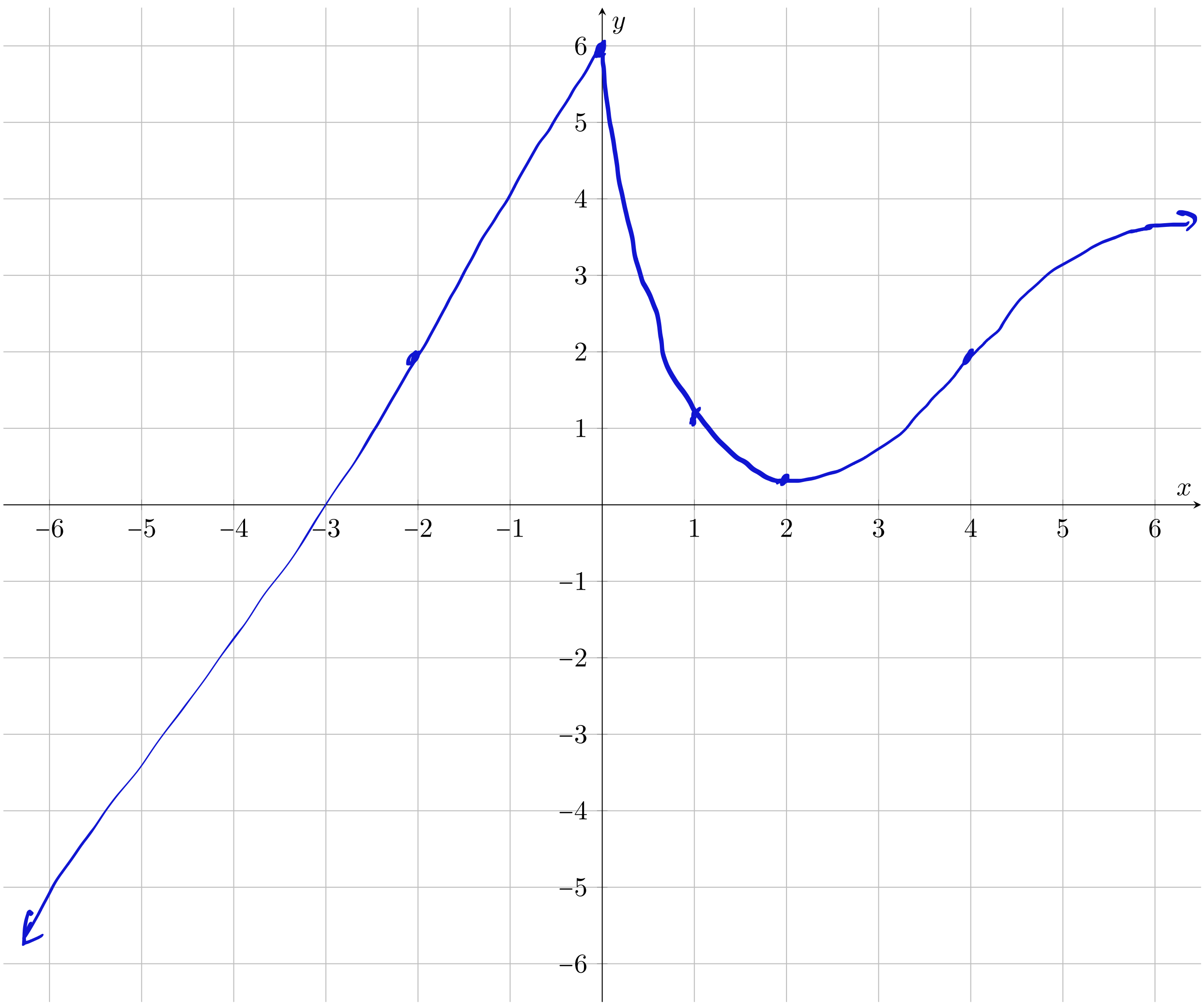
Solution: Let's use the values in the table to look at how the average rates of change change.

$$\frac{1.3 - 0.5}{64 - 61} = \frac{.8}{3} = 0.2\overline{6} \qquad \frac{2.3 - 1.3}{66 - 64} = \frac{1}{2} = 0.5 \qquad \frac{3.3 - 2.3}{67 - 66} = 1 \qquad \frac{7 - 3.3}{68 - 67} = 3.7$$

Since these average rates of change are increasing, that indicates that the number of additional hours for each additional degree Fahrenheit of heating is getting larger.

4. [7 points] On the axes below, sketch a graph of a single *function* $F(x)$ with domain including $-6 < x < 6$ that meets all of the following criteria:

- On the domain $[0, 2]$ the function is given by $F(x) = 6(0.2)^x$.
- $F(x)$ has an average rate of change of 0 between $x = -2$ and $x = 4$.
- $F(x)$ is increasing from $x = -6$ to $x = 0$.
- $F(x)$ is *not* invertible on the domain $0 < x < 6$.



many different graphs are possible

5. [11 points] A farmer is planning to plant a small apple orchard. She knows that the more trees she plants, the fewer apples each tree will produce due to the effects of crowding. In particular, if she plants x trees, when they are fully grown she expects each tree to produce a kilograms (kg) of apples each year, where

$$a = 20 - \frac{1}{5}x.$$

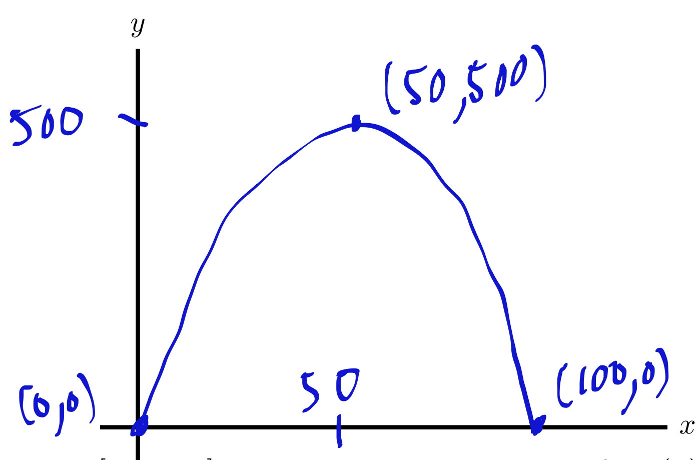
a. [2 points] Describe the meaning of the slope of this line in the context of this problem.

Solution: For each additional tree planted, the expected production from each tree will decrease by $\frac{1}{5}$ kilograms.

Once the trees are fully grown, the total yearly harvest of the farmer's orchard, in kilograms, will be given by the quadratic function

$$h(x) = x\left(20 - \frac{1}{5}x\right).$$

b. [5 points] Sketch a graph of $y = h(x)$ on the axes below. Be sure that the scale of each axis is clear, and that the vertex, vertical intercept, and any zeroes are clear. Also write the (x, y) coordinates of these points in the included blanks.



vertical intercept = $(0, 0)$

zeroes = $(0, 0)$ **and** $(100, 0)$

vertex = $(50, 500)$

c. [2 points] What is a reasonable domain for $h(x)$ given the context of the problem? Briefly explain.

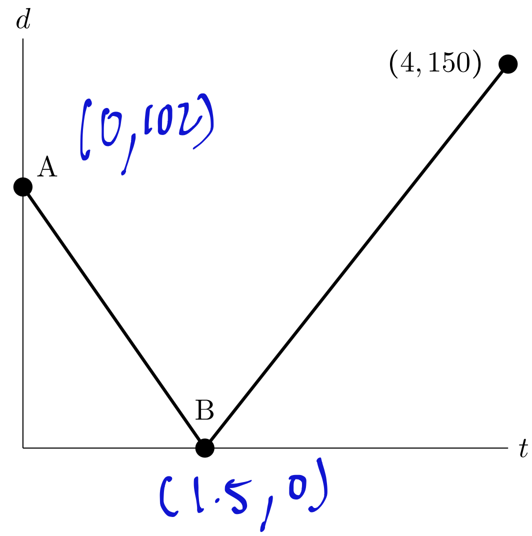
Solution: A reasonable domain for is function is $[0, 100]$. It doesn't make sense to produce *less* than 0 trees. And once we're at more than 100 trees, the total apple production is negative, which doesn't make sense.

d. [2 points] How many trees should the farmer plant to maximize her harvest? Briefly explain.

Solution: The farmer should plan 50 trees to maximize her harvest. We can see in our graph of $h(x)$ that the highest total production is achieved at the vertex of the parabola, which is at $(50, 500)$.

6. [9 points] The typical driving route from Ann Arbor to Chicago passes through the city of Kalamazoo. When UM students Elena and Victor recently drove from Ann Arbor home to Chicago, they began their drive traveling at a constant speed. As they passed Kalamazoo, they switched to a different constant speed for the rest of their trip.

Let $d = k(t)$ be Elena's and Victor's distance from Kalamazoo, in miles, t hours after beginning the trip. Part of the graph of $k(t)$, and part of the formula for $k(t)$, are given below.



$$k(t) = \begin{cases} 102 - 68t & 0 \le t < 1.5 \\ 60(t - 1.5) & \mathbf{1.5} \le t \le \mathbf{4} \end{cases}$$

a. [2 points] On the graph above, clearly label the points A and B with their (t, d) coordinates.

See above

b. [3 points] Fill in the missing parts of the formula for $k(t)$ given above. (It should match the graph of the function given.)

c. [4 points] Use the information given in this problem to answer the following.

How many miles is it from Ann Arbor to Kalamazoo?

Answer: **102**

How many hours does it take Elena and Victor to drive from Kalamazoo to Chicago?

Answer: **2.5**

At what speed is their car traveling between Ann Arbor and Kalamazoo? *Include units.*

Answer: **68 mph**