Math 105 — Second Midterm — November 14, 2023

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. Use a pencil for "bubble-in" questions so that you can easily erase your answer if you change your mind.
4. This exam has 13 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
6. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
7. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
8. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
9. You must use the methods learned in this course to solve all problems.
10. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2}+\cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	9	
2	6	
3	15	

Problem	Points	Score
4	7	
5	13	
6	10	
Total	60	

1. [9 points] The function $h(r)$ is **even** and **periodic** with period 6. Some values of $h(r)$ are given in the table below.

r	-2	0	1	3
$h(r)$	1.5	1	0	-1

The function $f(x)$ is given by $f(x) = \log(3x)$.

a. [5 points] Evaluate each of the following using information you know about $h(r)$; or if there is not enough information to so so, write NEI. *Write your answer in exact form, or rounded to two decimal places. Where relevant, show all work.*

(i.) $h(2) =$ 1.5

Solution: Because $h(r)$ is an even function, $h(2) = h(-2) = 1.5$.

(ii.) $h(h(0)) =$ $h(1) = 0$

(iii.) $h(10) =$ 1.5

Solution: Because $h(r)$ is a periodic function with period 6, $h(10) = h(10-12) = h(-2) = 1.5$.

(iv.) $f^{-1}(h(-2)) =$ $\frac{10^{1.5}}{3} \approx 10.541$

Solution: Since $h(-2) = 1.5$, we need to fine $f^{-1}(1.5)$. This is the same as solving for the x such that $f(x) = 1.5$.

$$\begin{aligned} \log(3x) &= 1.5 \\ 3x &= 10^{1.5} \\ x &= \frac{10^{1.5}}{3} \end{aligned}$$

b. [2 points] Is $h(t)$ invertible? Explain why it definitely is, why it definitely isn't, or if there isn't enough information to tell.

Solution: If $h(r)$ is even, then so is $h(t)$— the input variable doesn't change that. Thus $h(2) = h(-2)$. This means it doesn't pass the horizontal line test, and thus is not invertible.

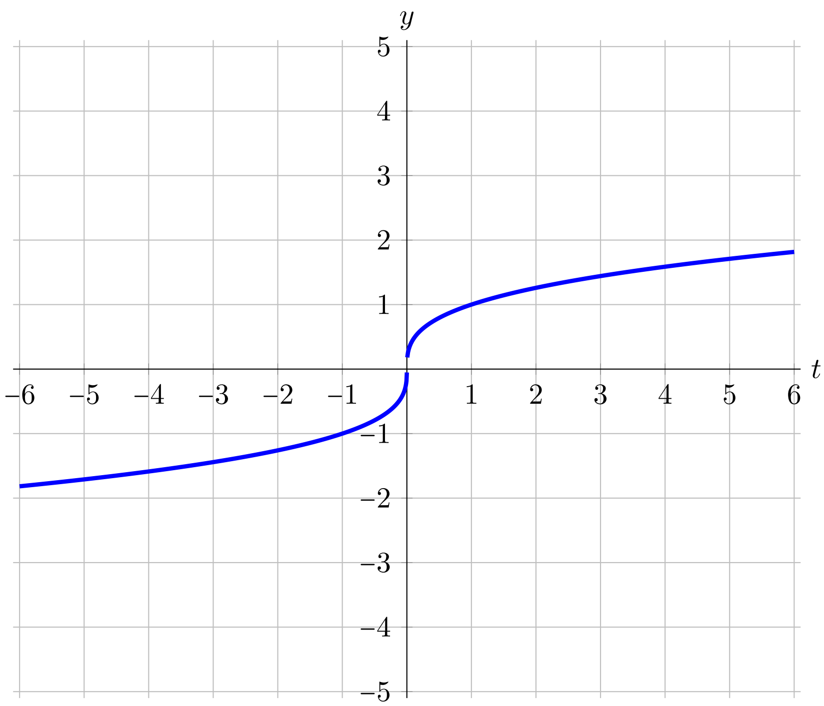
c. [2 points] Without using a calculator, Laurel claims that $f(33)$ is approximately 2. Explain how she could have known this!

Solution: $f(33) = \log(3 \cdot 33) = \log(99) \approx \log(100) = 2$.

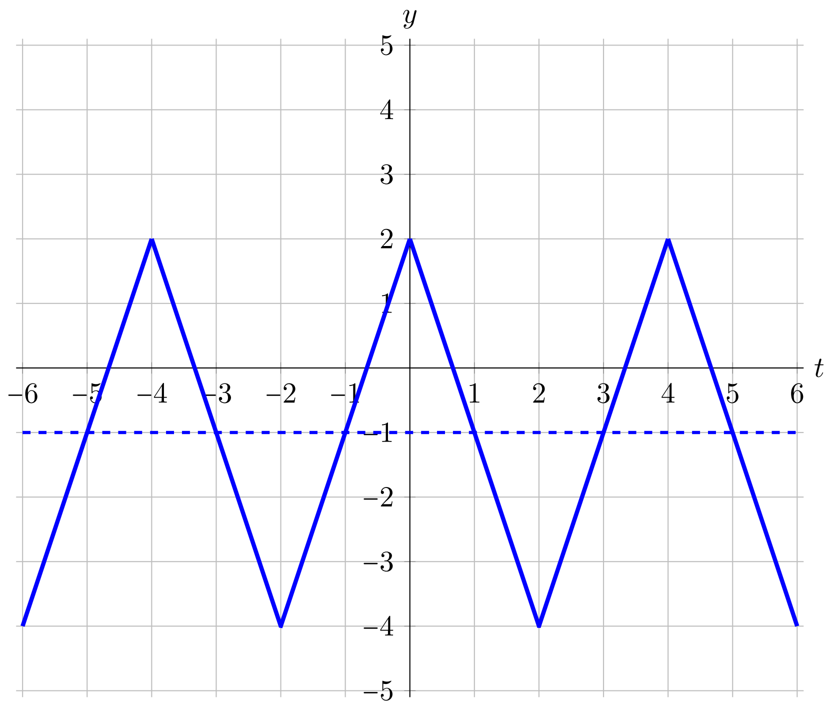
2. [6 points] For each of the following, sketch a graph of a function meeting the stated criteria, or explain why no such function exists.

Solution: There are many possible graphs satisfying the constraints given. Below are but two examples.

a. [3 points] An odd, invertible function defined on the domain $[-6, 6]$



b. [3 points] A periodic function with period 4, amplitude 3, and midline $y = -1$. *Include at least three periods in your sketch.*



y
t
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−5
−4
−3
−2
−1
1
2
3
4
5
6

3. [15 points] A scientist is observing two different ant colonies under different experimental conditions. From her data, it looks like

- Colony A's population increases by 10% every two hours.
- Colony B's population decreases by 7% every hour.

a. [1 point] By what factor is Colony A's population multiplied each hour? *Give your answer in exact form or rounded to two decimal places.*

a factor of: $\sqrt{1.1} \approx 1.049$

b. [2 points] What is the *continuous* decay rate of Colony B per hour as a percentage? *Give your answer in exact form or rounded to two decimal places.*

Solution: A 7% decay rate means we have a growth factor of 0.93. So we need to find the value of k such that $e^k = 0.93$. This is equivalent to $k = \ln(0.93) \approx -.07257$. Since we are asked for the *decay* rate, we should give the positive version, and give it as a percent: $|\ln(0.93)| \times 100$ % or$-\ln(0.93) \times 100$ % or 7.257%.

$-\ln(0.93) \times 100 \approx 7.257$ %

c. [2 points] How long will it take for Colony B to reach 25% of its original size? *Show all work. Give your answer in exact form or rounded to two decimal places.*

Solution: We need to solve for t:

$$\begin{aligned} Q_0 0.93^t &= 0.25 Q_0 \\ 0.93^t &= 0.25 \\ \ln(0.93^t) &= \ln(0.25) \\ t \ln(0.93) &= \ln(0.25) \\ t &= \frac{\ln 0.25}{\ln 0.93} \end{aligned}$$

$\frac{\ln 0.25}{\ln 0.93} \approx 19.103$ hours

d. [4 points] If Colony A starts with 1000 ants and Colony B starts with 10,000 ants, after how many hours will the colonies have equal populations? *Show all work. Give your answer in exact form or rounded to two decimal places.*

Solution: A formula for the number of ants in Colony A is $1000 \cdot (\sqrt{1.1})^t$. A formula for the number of ants in Colony B is $10000 \cdot 0.93^t$. We want to find the value of t that makes these two functions equal. That is, we need to solve the following for t:

$$\begin{aligned}
1000 \cdot (\sqrt{1.1})^t &= 10000 \cdot 0.93^t \\
\left(\frac{\sqrt{1.1}}{0.93}\right)^t &= 10 \\
\log\left(\frac{\sqrt{1.1}}{0.93}\right)^t &= \log 10 \\
t \log\left(\frac{\sqrt{1.1}}{0.93}\right) &= 1 \\
t &= \frac{1}{\log\left(\frac{\sqrt{1.1}}{0.93}\right)} \approx 19.152
\end{aligned}$$

Another way to solve the same starting equation is shown below. Note that the final answers will look different. But we can either use a calculator or log identities to see that they are actually equivalent.

$$\begin{aligned}
1000 \cdot (\sqrt{1.1})^t &= 10000 \cdot 0.93^t \\
\ln(1000 \cdot (\sqrt{1.1})^t) &= \ln(10000 \cdot 0.93^t) \\
\ln(1000) + \ln(\sqrt{1.1})^t) &= \ln(10000) + \ln(0.93^t) \\
\ln(1000) + t\ln(\sqrt{1.1})) &= \ln(10000) + t\ln(0.93) \\
t\ln(\sqrt{1.1})) - t\ln(0.93) &= \ln(10000) - \ln(1000) \\
t\left(\ln(\sqrt{1.1}) - \ln(0.93)\right) &= \ln(10000) - \ln(1000) \\
t &= \frac{\ln(10000) - \ln(1000)}{\ln(\sqrt{1.1}) - \ln(0.93)} \approx 19.152
\end{aligned}$$

$\frac{1}{\log\left(\frac{\sqrt{1.1}}{0.93}\right)} \approx 19.152$ hours

(Problem continues on the next page.)

The scientist now observes two additional different ant colonies. From her data, it looks like

- Colony C's population doubles for the first time after 2.5 hours; doubles again 1.5 hours after that; then doubles a third time 1 hour after that.
- Colony D's population is given by $P = D(t) = 1200 - 300e^{-0.11t}$, where P is the number of ants and t is measured in hours since the experiment started.

e. [2 points] Is Colony C growing exponentially? Circle your answer below. If YES, find its growth factor. If NO, explain why not.

YES NO

Explanation or Growth Factor:

Solution:
NO.

Any exponentially growing function should have a *constant* doubling time. Since the doubling time of this function is changing, it cannot be growing exponentially.

f. [4 points] Find a general formula $D^{-1}(P)$ and explain what that function means. *Show all work.*

Solution: We need to solve $P = 1200 - 300e^{-0.11t}$ for t, which will give us t as a function of P— in other words, our inverse function.

$$\begin{aligned} P &= 1200 - 300e^{-0.11t} \\ P - 1200 &= -300e^{-0.11t} \\ \frac{P-1200}{-300} &= \frac{1200-P}{300} = e^{-0.11t} \\ \ln\left(\frac{1200-P}{300}\right) &= -0.11t \\ t &= \frac{1}{-0.11}\ln\left(\frac{1200-P}{300}\right) \end{aligned}$$

$D^{-1}(P) =$ $\frac{1}{-0.11}\ln\left(\frac{1200-P}{300}\right)$

Meaning of $D^{-1}(P)$:

Solution: $D^{-1}(P)$ gives us the number of hours after the experiment started at which there are P ants in Colony D.

4. [7 points] On a warm fall day, Schinella decides to walk home from work. Let $d = f(t)$ be the function giving Schinella's distance **from work**, in miles, t minutes after she leaves work.

a. [3 points] Her walk home from work is 3 miles. Schinella wants to write a new function $g(h)$ that gives her distance **from home**, in miles, h **hours** after she leaves work. Write a formula for $g(h)$ in terms of f.

$g(h) =$ $3 - f(60h)$

b. [2 points] Schinella (who is from Canada) wants to write another new function $k(t)$ that gives her distance from work in **kilometers** t minutes after she leaves work. Given that 1 mile is about 1.6 kilometers, circle the correct formula for $k(t)$ below.

$\boxed{1.6f(t)}$ $\qquad f(1.6t)$ $\qquad \dfrac{1}{1.6}f(t)$ $\qquad f\left(\dfrac{t}{1.6}\right)$

c. [2 points] Let $c(t)$ be the function that gives the number of episodes of the podcast *Canadaland* that Schinella has listened to in the first t minutes of her walk. Assume that both $c(t)$ and $f(t)$ are invertible. Using those functions or their inverses, write an expression for Schinella's distance from work, in miles, after she's listened to 2.5 episodes of *Canadaland* while walking home.

$f(c^{-1}(2.5))$ miles

5. [13 points]

a. [4 points] A zookeeper has determined that the function $w(t)$ below provides a good model of the weight, in ounces, of a certain kind of snake t years after it hatches.

$$w(t) = -2e^{-(t-16)/5} + 52$$

Find the value of each of the following **as numbers rounded to two decimal places**. Then **briefly interpret** what each quantity means in the context of the problem.

i. $w(0) =$ ≈ 2.93 **Meaning:** When the snake is first born, it weighs approximately 2.93 ounces.

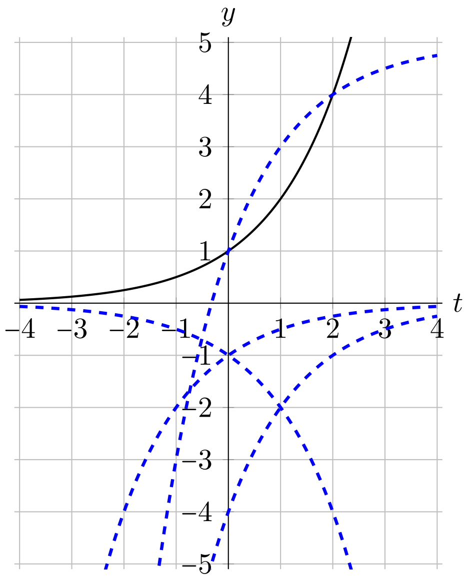
ii. $\lim_{t\to\infty} w(t) =$ 52 **Meaning:** As the snake ages, its weight gets closer and closer to 52 ounces.

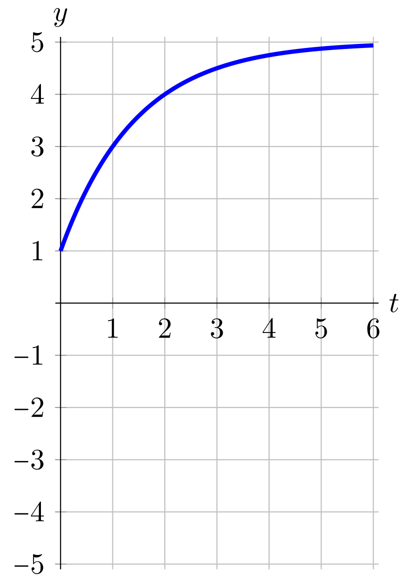
(Problem continues on the next page.)

b. [2 points] The zookeeper also has a model $\ell(t)$ of the length, in feet, of this type of snake t years after it hatches.

$$\ell(t) = -2^{-(t-2)} + 5$$

Using the graph of $y = 2^t$ below as a starting point, sketch a graph of $y = \ell(t)$, for $0 \le t < 6$, on the axes provided to the right.





Solution: The graph of the left shows sketches of the tranformations done, in order, to the original graph as a way of arriving at the final graph. One could also plug in points such as $t = 1$, $t = 2$, etc., to get a sense for the final graph.

c. [5 points] List the transformations you need to apply to the graph of $y = 2^t$ to transform it to that of $y = \ell(t)$. Fill in the first blank with one of the phrases below. Fill in the second blank with a number, "by a factor of" and a number, or N/A for reflections.

SHIFT IT TO THE LEFT	STRETCH IT HORIZONTALLY	REFLECT IT ACROSS THE y-AXIS
SHIFT IT TO THE RIGHT	COMPRESS IT HORIZONTALLY	REFLECT IT ACROSS THE t-AXIS
SHIFT IT UP SHIFT IT DOWN	STRETCH IT VERTICALLY	COMPRESS IT VERTICALLY

First, **Reflect it across the t-axis** by **N/A**

then, **Reflect it across the y-axis** by **N/A**

then, **Shift it to the right** by **2**

then, **Shift it up** by **5**

Note that there are multiple orders this could be listed it. What's important is that the reflection across the t-axis is listed before the shift up; and that the reflection across the y-axis is listed before the shift right. If ether shift is listed *first*, then the direction of the shift would have to be reversed from what is shown here in order to account for the reflection coming second.

Also note that there is a wholly other way to think about this problem. Because $-2^{-(t-2)}+5$ is algebraically equivalent to $-2^{-t}2^4+5$ the horizontal shift is actually equivalent to a vertical stretch by 16. Any students who gave a vertical stretch instead of a horizontal shift would also have received full credit.

d. [2 points] Give equations for all vertical and horizontal asymptotes of $\ell(t)$. If there are none, write NONE.

Vertical Asymptotes: **NONE**

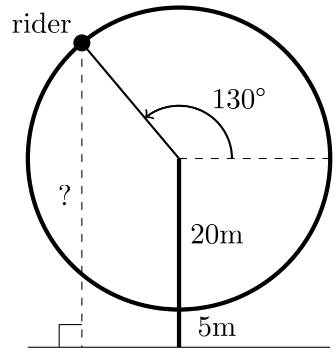
Horizontal Asymptotes: $y = 5$

Solution: Because this is a shifted exponential function, there are no vertical asymptotes, only horizontal ones. We can see that as we plug in larger and larger values of t, that the term $-2^{-(t-2)}$ gets smaller and smaller, approaching 0, because we are raising 2 to higher and higher negative powers. So overall the function will approach 5.

Another way to see this is by looking at how the original asymptote $y = 0$ is transformed under the transformations in the list above. The only one that affects it is the final shift up by 5, make an asymptote of $y = 5$.

6. [10 points] A diagram showing a Ferris wheel is below. The radius of the Ferris wheel is 20 meters, and the lowest point (where people board) is reached by a small set of stairs and is 5 meters above ground level. We'll consider the following question about this scenario:

When an arm of the Ferris wheel is making an angle of 130° *with horizontal, how high is that rider off the ground?*



a. [3 points] Ayisha really likes to do trig problems using a unit circle perspective. She comes up with a correct answer that involves $\sin(130^\circ)$ in her answer. What is Ayisha's answer? *Give your answer in exact form.*

Solution: The vertical distance of the rider above the center of the Ferris wheel would be $\sin(130^\circ)$ *if* we were in the unit circle. But because this circle is scaled up by 20m, their height above the center of the Ferris Wheel level woud be $20 \cdot \sin(130^\circ)$. To get all the way to the ground, we need to add the distance from the Ferris wheel center to the ground, which is 25 meters. This gives us our final answer below.

Another way to think of this is using the formula for a point on a circle of radius r centered at point (x, y). In this case, we are looking for a y-coordinate, so the formula would give us the same thing: $20 \cdot \sin(130^\circ) + 25$.

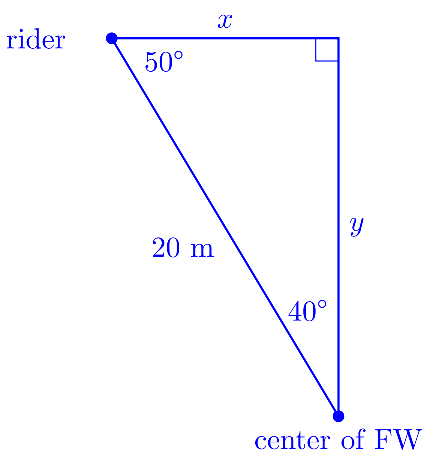
$20\sin(130^\circ) + 25$ meters

b. [2 points] Oh no! Ayisha accidentally had her calculator in radians when she computed her answer. What would she get in that case? How could she recognize right away that her answer was incorrect?

Solution: If you put the expression above into the calculator when your calculator is in radians, then you get approximately 6.398. This is clearly wrong because the rider should be above 25m when they're at 130°.

c. [4 points] Bruno really likes to do trig problems using right triangles. He comes up with a correct answer that involves $\cos(40^\circ)$. Draw the right triangle Bruno could have been considering and use that triangle to find Bruno's correct expression for the rider's height from the ground. *Give your answer in exact form.*

Solution: Here is a diagram of a triangle extracted from the Ferris wheel picture, with known and unknown lengths and angles shown.



We want to know the value of y in the picture, so we can use either 40° or 50° to find it. Since we're told Bruno used 40°, we'll use that!
Then we know that $\cos(40°) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{y}{20}$. We can solve for y and get $y = 20\cos(40°)$.

(Another way to see is this is to imagine rotating the triangle 90° clockwise. Then we can think of 40° as an angle above the x-axis, and the length labeled y as a horizontal (cosine) measurement in a triangle scaled up to a radius of 20m.)

Adding this value for y to the 25 meter height of the center of the Ferris wheel, we get

$$20\cos(40°) + 25$$

as the riders height off of ground level.

$20\cos(40°) + 25$ meters.

d. [1 point] Using a calculator, verify that Ayisha's and Bruno's answers agree. That is, find the numerical value of both expressions. *(We're now assuming that Ayisha's calculator is correctly back in degrees!)*

Solution: Both expressions evaluate to approximate 40.32. This also makes sense numerically because it's between 25 and 45; further, visually we can see that it should be closer to 45 than to 25.