Math 105 — Final Exam — December 8, 2023

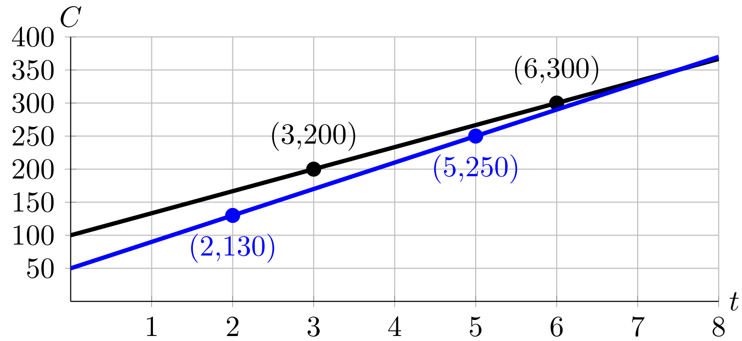
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. Use a pencil for "bubble-in" questions so that you can easily erase your answer if you change your mind.
4. This exam has 17 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
5. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
6. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
7. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
8. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
9. You must use the methods learned in this course to solve all problems.
10. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
11. Include units in your answer where that is appropriate.
12. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
13. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	11	
2	7	
3	8	
4	10	
5	11	

Problem	Points	Score
6	10	
7	7	
8	8	
9	8	
Total	80	

1. [11 points] Pam the Plumber charges customers $P(t)$ dollars for a t hour visit to their house. A graph of $C = P(t)$ is shown below.



a. [2 points] Find the value of $P(0)$ and explain what it means in context.

$P(0) =$ **100**

Meaning: This is how much Pam charges to arrive at house, without having done any work yet.

b. [2 points] Find the slope of $P(t)$ and explain what it means in context.

Slope: 100/3

Meaning: After her arrival cost, for every 3 hours Pam stays, she charges \$100. Or, she charges \$33.3 / hour.

c. [2 points] Beth the Bathtub-fixer charges \$50 for a home visit plus \$40/hour spent there. Let $B(t)$ be the amount she charges for a t hour visit. Add the graph of $C = B(t)$ for $0 \le t \le 8$ to axes above. **Clearly label at least two points on your graph with their (t, C) coordinates.**

Solution: See graph above.

d. [5 points] Find the coordinates of the point where the two graphs intersect and explain what this means in context. *Show all work. Give your final answer in exact form, or rounded to at least two decimal places.*

Solution: We need to find where the two linear equations intersect. To do this we can solve the following equation for t:

$$50 + 40t = 100 + 100t/3$$

By adding and subtracting terms from both sides, this simplifies to:

$$20t/3 = 50$$

which finally is equivalent to:

$$t = 150/20 = 7.5$$

To find the C-coordinate, we can plug $t = 7.5$ into either expression. In particular:

$$50 + 40 \times 7.5 = 350$$

Point $(t, C) =$ $(7.5, 350)$

Meaning: This means that when Pam and Beth each work 7.5 hours, they earn the same amount of money for the visit.

2. [7 points] The amount, in milligrams (mg), of a certain drug in a patient's bloodstream t minutes after it is administered is given by:

$$V(t) = 120e^{-0.006t}$$

a. [2 points] By what percentage does the amount of the drug in the patient's bloodstream decrease each minute? *Show all work. Give your answer in exact form, or rounded to at least* ***three*** *decimal places.*

$100(1 - e^{-0.006}) = 0.598$ %

b. [3 points] How long does it take for the amount of the drug in the patient's bloodstream to decrease to 10 mg? *Show all work. Give your answer rounded to the* ***nearest minute***.

Solution: We need to solve for the time t at which

$$120e^{-0.006t} = 10$$

Once we are past that time, the drug will be less than 10mg.

$$\begin{aligned} 120e^{-0.006t} &= 10 \\ e^{-0.006t} &= \frac{1}{12} \\ -0.006t &= \ln(\tfrac{1}{12}) \\ t &= \ln(\tfrac{1}{12})/-0.006 \approx 414 \end{aligned}$$

$\ln(\frac{1}{12})/-0.006 \approx 414$ minutes

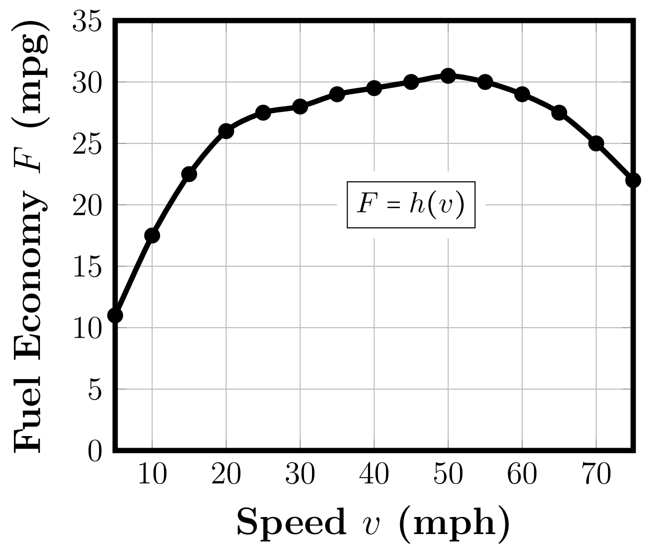
c. [2 points] The amount, in mg, of a *different* drug in a patient's bloodstream t minutes after it is administered is given by $G(t)$. Some values of $G(t)$ are given below. Could $G(t)$ be exponential? *Show all work.*

t, in minutes	20	30	50
$G(t)$, in mg	95	76	48.64

Solution: There are several ways that we could check if this is exponential. One way is to imagine a theoretical output for $t = 40$ minutes. If this function were exponential, then the multiplicative increase from $G(20)$ to $G(30)$ should be the same as the multiplicative increase from $G(30)$ to $G(40)$ and from $G(40)$ to $G(50)$. We can compute that the multiplicative increase from $G(20)$ to $G(30)$ is 76/95. If we apply that multiplicative increase to $G(30) = 76$ we get $76^2/95$ as a hypothetical value for $G(40)$. If we apply it one more time we get $76^3/95^2 = 48.64$ That is, this is the value we'd expect for $G(50)$ if our function were exponential. Since that is exactly the value for $G(50)$ we see in our table, we can conclude that our function could, in fact, be exponential.

(Circle One) COULD BE EXPONENTIAL NOT EXPONENTIAL

3. [8 points] The website `fueleconomy.gov` gives the graph on to the right to show fuel economy (in miles per gallon or mpg) as a function of speed (in miles per hour or mph) for a particular type of car. We'll call the function defined by this graph $h(v)$.



a. [1 point] What can you say about the concavity of $h(v)$ over the domain $[10, 30]$? *Circle one answer; no explanation necessary.*

CONCAVE UP

$\boxed{\text{CONCAVE DOWN}}$

NEITHER

b. [1 point] $h(v)$ is not an invertible function. Explain in 1–2 sentences how we know.

Explanation: Since the graph does not pass the horizontal line test, it cannot be invertible. That is, there is an output value with two different input values, so the function cannot be inverted.

c. [1 point] If we wanted to restrict the domain of $h(v)$ so that is *was* an invertible function, what would be a good domain to use?

Domain: $[5, 50]$ **or** $[50, 75]$

Another function $c(F)$, which is invertible, gives the cost of gas in dollars per mile when we have a fuel economy of F mpg.

d. [4 points] Write a sentence or phrase that gives the meaning of each of the following equations or expressions. Or, if it does not make sense in context, explain why not.

i. $c(h(80)) = 0.22$ When this particular car is going 80 miles per hour, the cost of gas is 0.22 dollars per mile.

ii. $c^{-1}(0.18)$ This is the fuel economy (in mpg) when the cost of gas is 0.18 dollars per mile.

e. [1 point] In this context, what is a reasonable domain for $c(F)$? *No explanation necessary.*

Domain: $[0, 50]$

Solution: There are many different ways we could give a reasonable domain for c. One thing we know is that it shouldn't include negative numbers, since a negative fuel economy doesn't make sense. On the other hand, we know that any domain for c should include at least the range of fuel economy we see in the graph of $F = h(v)$. In particular, it should include at least $[11, 30]$.

4. [10 points] Below are several quadratic equations, labeled A – J:

A. $y = -3x(x-2)$

B. $y = x(x+2)$

C. $y = x^2 + 1$

D. $y = x^2 - 1$

E. $y = 3x(x+2)$

F. $y = (x-1)(x+1)$

G. $y = -3(x-1)^2 + 3$

H. $y = (x-1)(x+1) + 2$

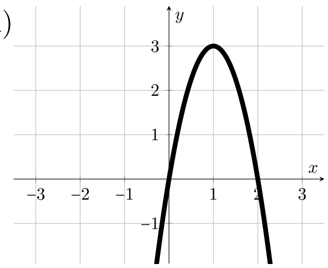
I. $y = -x(x-2) + 2$

J. $y = (x-1)^2 + 1$

a. [8 points] For each of the graphs below list ALL equations (among A–J) that match the given graph. If no equations match the graph, write NONE.

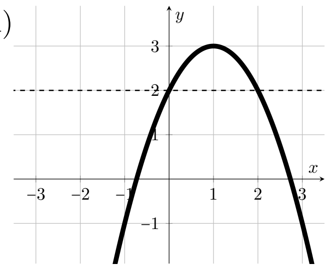
Note that some graphs have dashed lines included, to help you think of those graphs as *shifts* of other quadratic equations.

(i)



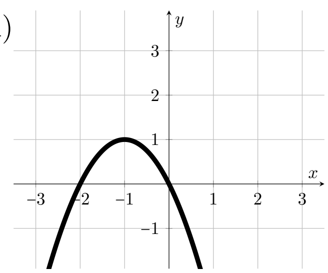
Equation(s): A, G

(iii)



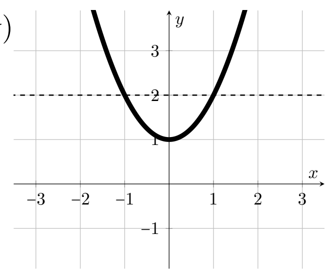
Equation(s): I

(ii)



Equation(s): NONE

(iv)



Equation(s): H, C

b. [2 points] It turns out we can also think of these graphs as transformations of each other. Claudia claims that we can transform the graph of (ii) into the graph of (i) using a vertical stretch followed by horizontal shift. She is right! Find the stretch factor and shift value that will complete that transformation:

Stretch vertically by a factor of: 3

Shift right by: 2

5. [11 points] The water levels in a large bay fluctuate due to the tides. High tide, which is when water levels are at their maximum, happens roughly twice per day. Similarly, low tide is when water levels are at their minimum.

a. [6 points] At one particular location in this bay, the depth, in feet, of the water t hours after midnight on December 1 was given by

$$D(t) = 11\cos\left(\frac{24\pi}{149}(t-3)\right) + 56.$$

i. What is the depth of the water, in feet, at this location at high tide? At low tide?

$56 + 11 = 67$ feet at high tide

$56 - 11 = 45$ feet at low tide

ii. Find the period of $D(t)$, either in exact form or rounded to two decimal places. Then interpret what it means in the context of this problem.

Period: $149/12 \approx 12.42$

Meaning:

This is the time, in hours, between high tides. (Equivalently, the time, in hours, between low tides.)

iii. Find the times t of all high tides that occur on December 1. *Give your answer as a list of t-values in exact form or rounded to two decimal places.*

Solution: The graph of the cosine function achieves its maximum value at $t = 0$ and then every period to the left and right of that. Since this is a cosine graph that's been shifted right by 3, it will achieve its maximum at $t = 3$ plus every period to the left and right of that.

$t =$ $3,\ 3 + 149/12 \approx 15.42$

This problem continues on the next page.

b. [5 points] At another location in the bay, the depth, in feet, of the water t hours after midnight on December 1 was given by

$$P(t) = 9\sin\left(\frac{24\pi}{149}t\right) + 40.$$

Find the t-values of all times on December 1 that the water level at this location was 45 feet. *Give your answer as a list of t-values in exact form or rounded to two decimal places.*

Solution: We can start by fining one solution, then using symmetry to find additional solutions. Algebraically, we can find our first solution as follows:

$$\begin{aligned} 45 &= 9\sin\left(\frac{24\pi}{149}t\right) + 40 \\ 5 &= 9\sin\left(\frac{24\pi}{149}t\right) \\ \frac{5}{9} &= \sin\left(\frac{24\pi}{149}t\right) \\ \arcsin\left(\frac{5}{9}\right) &= \frac{24\pi}{149}t \\ \frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right) &= t \end{aligned}$$

We can use a calculator to help us see that this is approximately equivalent to 1.16 hours, meaning that this is one of the solutions we're looking for on December 1, about 1.16 hours after midnight.

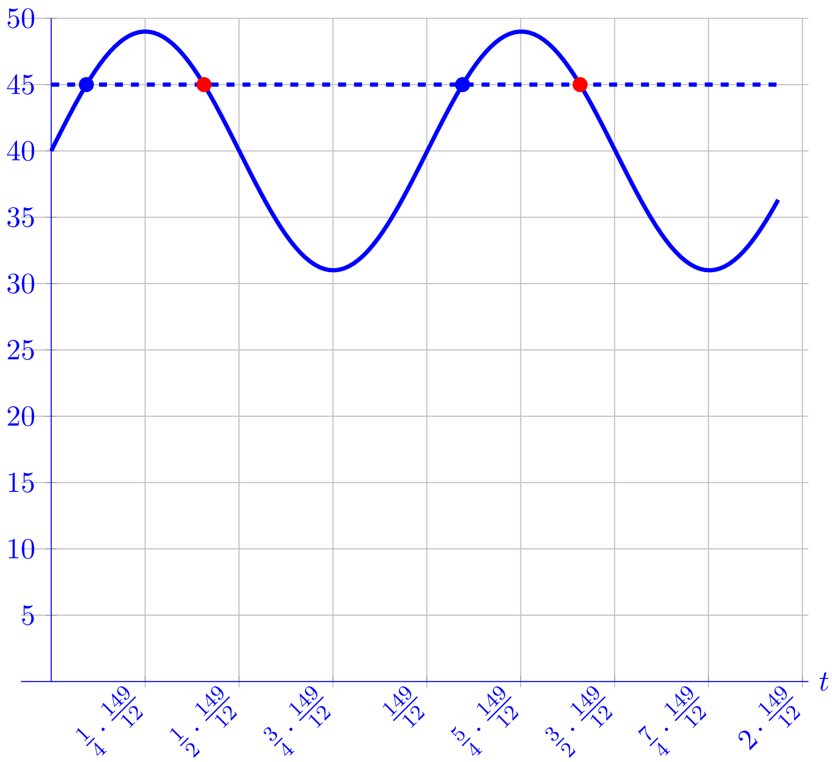
The period of this function is the same as in the earlier part of this problem: $2\pi \div \frac{24\pi}{149} = \frac{149}{12}$. Knowing this, we'll have a second solution one period later at

$$t = \frac{149}{24\pi}\arcsin\left(\frac{5}{9}\right) + 149/12 \approx 13.58$$

However, this is not all of our solutions. The water level passes 45 feet both on its way up to high tide, and again on its way back down to low tide. To find the additional solutions we can use the symmetries of a sin graph (or of a circle). Using the period, midline, amplitude, etc., we can sketch the graph below and see where it intersects the line $y = 45$.

Using a calculator, we can see that the solutions we've found so far correspond to the two blue dots (1st and 3rd dots going left to right). The two we're missing are the red dots below (2nd and 4th dots going left to right).

Solution:



To find the t-values of the red dots, we can notice that the first red dot is as far to the *left* of a half period, as the first blue dot is to the *right* of 0. In other words, the t-value of the first red dot (2nd dot from the left) will be:

$$\frac{1}{2} \cdot \frac{149}{12} - \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right) \approx 5.044$$

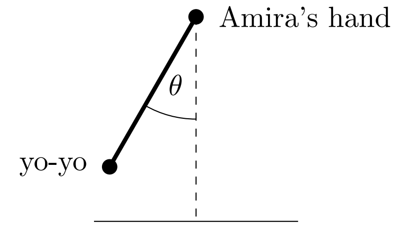
And our final solution will be one period later:

$$\left(\frac{1}{2} \cdot \frac{149}{12} - \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right)\right) + \frac{149}{12} \approx 17.461$$

$$t = \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right),\ \frac{1}{2} \cdot \frac{149}{12} - \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right),\ \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right) + \frac{149}{12}, \frac{1}{2} \cdot \frac{149}{12} - \frac{149}{24\pi} \arcsin\left(\frac{5}{9}\right) + \frac{149}{12}$$

Numerically: $t \approx 1.164, 5.004, 13.581, 17.461$

6. [10 points] Amira is using a yo-yo as a pendulum by holding the string and letting the yo-yo swing back and forth in a plane—that is, just left to right, not making any kind of ellipse when viewed from above. The symbol θ denotes the maximum angle the string makes with the vertical, as shown in the diagram to the right.



a. [2 points] If θ is 15° and the length of the string between Amira's hand and the yo-yo is 3 ft, what is the length of the entire arc that the yo-yo swings through as it travels left to right? *Show all work. Give your answer in exact form or rounded to at least two decimal places.*

Solution: Since $\theta = 15°$, then the angle of the swing from left to right is twice as much: 30°. This is 30/360 (or $\frac{1}{12}$) of a full rotation, so $2\pi \cdot \frac{30}{360}$ radians. Since the radius is 3ft, the total length of this arc is: $3 \cdot 2\pi \cdot \frac{30}{360} \approx 1.57$

Answer: $3 \cdot 2\pi \cdot \frac{30}{360} \approx 1.57$ ft

b. [1 point] If Amira adjusted the yo-yo so that the length of the string between her hand the yo-yo were only 1.5 feet instead, how would that change the length of the arc that the yo-yo swings through? *Show your work or explain.*

Solution: We'd have almost the same expression as above, but instead of 3 we'd have 1.5 being multiplied by $2\pi/12$. So our result would be half as long:

$$1.5 \cdot \frac{2\pi}{12} \approx 0.78$$

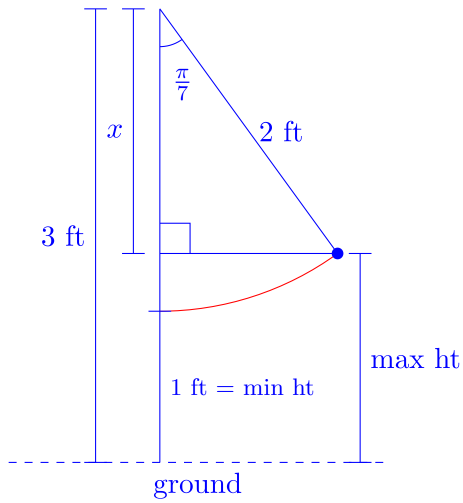
c. [7 points] Now suppose that

- the length of the string between Amira's hand and the yo-yo is 2 feet,
- at its lowest point, the yo-yo is 1 foot above the ground,
- θ is $\pi/7$ radians,
- and that it takes 1.6 seconds for the yo-yo to make a full swing from left to right and back to left again.

Give all answers below in exact form or rounded to two decimal places.

Find the maximum height of the yo-yo. *Show all work, **including a diagram**.*

Solution: Here is a diagram depicting what is described above.



If we can find the value of marked x in the diagram, then we can find the maximum height as $3 - x$. To find x we can use

$$\cos(\pi/7) = x/2$$

That is, $x = 2\cos(\pi/7) \approx 1.80$. So the maximum height is $3 - 2\cos(\pi/7) \approx 1.2$.

Answer: $3 - 2\cos(\pi/7) \approx 1.2$ ft

Let $h(t)$ be the function giving the height, in feet, of the yo-yo at time t seconds after it is released from its maximum height. Find the amplitude and period of $h(t)$.
Include units.

Solution: We found our maximum height above. To find the amplitude we can take (max ht − min ht)/2. Since the minimum height is 1 ft (at the bottom of the pendulum swing), this is:

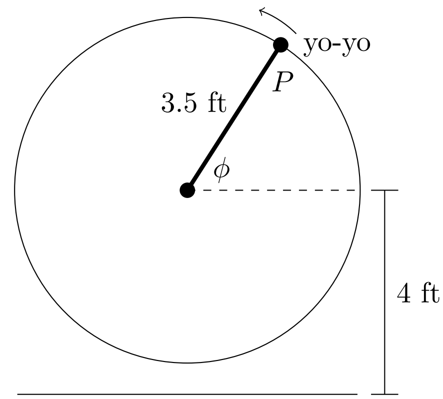
$$[(3 - 2\cos(\pi/7)) - 1]/2 \approx (1.2 - 1)/2 \approx 0.1$$

To find the amplitude we need to use the given fact that it takes 1.6 seconds for the pendulum to swing from left to right back to left again. However, if we are considering the *height* as a function of time, this starts repeating itself every half swing (right to left), so the period would be $1.6/2 = 0.8$ seconds.

Answer: amplitude = $((3-2\cos(\pi/7))-1)/2 \approx (1.2-1)/2 \approx 0.1$ **feet**

Answer: period = 0.8 **seconds**

7. [7 points]



Amira's friend Paul borrows her yo-yo and starts spinning it in a counterclockwise circle at a constant speed. His hand holds the string, at the center of the circle shown, 4 feet off the ground. The length of the string between his hand and the yo-yo is 3.5 feet.

In this problem, measure angles counterclockwise from the positive horizontal as usual. When the yo-yo is at point P, the angle ϕ as shown in the diagram to the right is $\frac{3\pi}{8}$ radians.

a. [3 points] How high off the ground is the yo-yo when it is at point P? *Give your answer in exact form or rounded to at least two decimals.*

Height: $3.5\sin(\frac{3\pi}{8}) + 4 \approx 7.23$ feet

b. [2 points] After the yo-yo travels most of the way around the circle from its current position, there will be a moment at which it is directly underneath point P. Find the angle, in radians, between 0 and 2π, at which this occurs. *Give your answer in exact form or rounded to at least two decimals.*

Angle: $2\pi - \frac{3\pi}{8}$ radians

c. [2 points] It takes the yo-yo 1 second to make a complete circle. What angle, in radians, will the yo-yo make with the positive horizontal 1/3 of a second after it is at point P? *Give your answer in exact form or rounded to at least two decimals.*

Solution: If it takes 1 second to make a full rotation (2π radians), in 1/3 of a second it will make one third of a rotation, or $2\pi/3$ radians. Since we are starting at $\frac{3\pi}{8}$ radians, we need to find one third of a rotation later, or

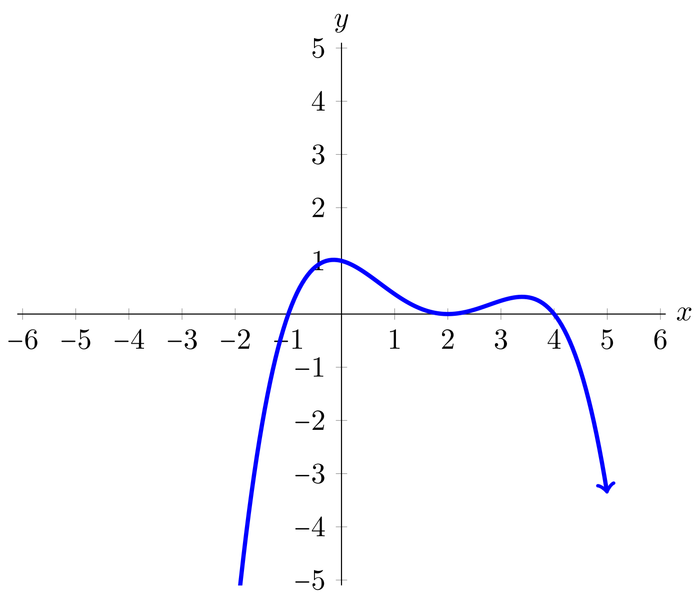
$$\frac{3\pi}{8} + \frac{2\pi}{3} = \frac{25\pi}{24} \approx 3.27$$

Angle: $\frac{3\pi}{8} + \frac{2\pi}{3} \approx 3.27$ radians

8. [8 points]

a. [4 points] Sketch a graph of a polynomial $f(x)$ satisfying the following conditions:

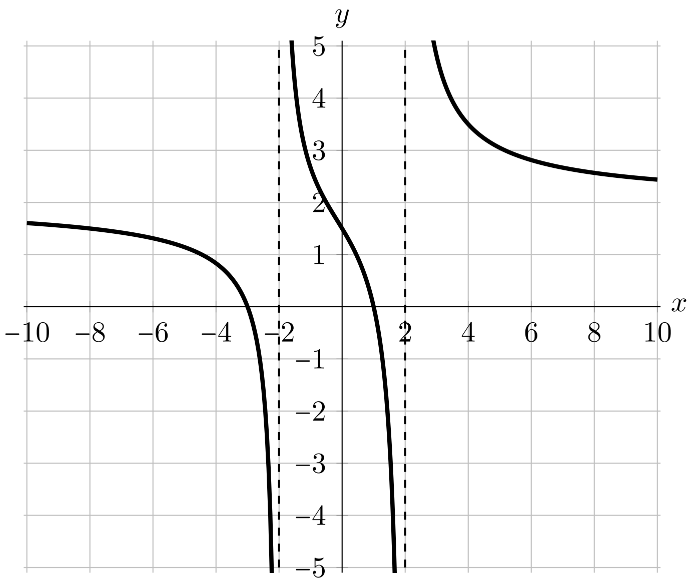
- $f(x)$ has zeros at $x = -1$, 2, and 4
- $\lim_{x \to -\infty} f(x) = -\infty$
- the y-intercept is 1
- $f(x)$ is of degree 4



Solution: There are two basic shapes that are possible here. For it to have the limit listed, a zero at $x = -1$, and vertical intercept at $y = 1$, then it must have a single zero at $x = -1$. However, for it to be degree four, it must have a multiplicity-2 zero at either $x = 2$ or $x = 4$. The graph shown above shows a multiplicity-2 zero at $x = 2$, but the other option is also possible.

b. [4 points] Write a possible formula for the graph of the rational function shown below. For clarity, its features are also described below.

- the y-intercept is 1.5
- the zeros are -3 and 1.
- horizontal asymptote of $y = 2$
- vertical asymptotes of $x = -2$ and $x = 2$



$y = \frac{2(x+3)(x-1)}{(x+2)(x-2)}$

Solution:
To begin with, since we have zerps of -3 and 1, we know we have factors of $(x+3)$ and $(x-1)$ in the numerator. Since we have vertical asymptotes at $x = -2$ and $x = 2$, we know we have factors of $(x+2)$ and $(x-2)$ in the denominator. As a preliminary function, we have so far:

$$\frac{(x+3)(x-1)}{(x+2)(x-2)}$$

However, we still need to account for the vertical intercept and horizontal asymptotes. We already have "matching degrees" in numerator and denominator, which will give us a non-zero horizontal asympote as desired. However, we want a horizontal asympote of $y = 2$, so we need the ratios of the leading coefficients to of numerator and denominator to be 2. We can edit our draft function above to get this:

$$\frac{2(x+3)(x-1)}{(x+2)(x-2)}$$

Finally, we should check that we have the right vertical intercept:

$$\frac{2(0+3)(0-1)}{(0+2)(0-2)} = \frac{-6}{-4} = 1.5$$

So we have met the last criteria as well!

9. [8 points]

a. [5 points] Find the values of the following limits. Your answer may be a numerical value, ∞, or $-\infty$. *You do not need to show work, but limited partial credit may be earned from work shown.*

(i) $\lim_{x\to 2} \dfrac{3(x-1)(x-2)}{(x-2)(x+3)} =$ ______________

Solution: This problem was removed from the exam.

(ii) $\lim_{x\to \infty} \dfrac{3(x-1)(x-2)}{(x-2)(x+3)} =$ **3**

Solution: Because both the numerator and denominator have degree 2, we need to look at the ratio of their leading coeffients to see what the value of the horizontal asymptote is. In this case, the leading coeffient of the numerator is 3 and the leading coefficient of the denonominator is 1, so the limit will approach $3/1 = 3$.

(iii) $\lim_{x\to \infty} \dfrac{x^8 - 7^x}{6^x + x^9} =$ $-\infty$

Solution: Because exponentially growing functions eventually dominate any polynomial, we need to only focus on 7^x and 6^x to determine the long-range behavior of this function. That is, the limit of the function given is the same as

$$\lim_{x\to\infty} \frac{-7^x}{6^x} = \lim_{x\to\infty} -\left(\frac{7}{6}\right)^x = -\infty$$

(iv) $\lim_{x\to\infty} \ln(x) =$ ∞

Solution: Even though the graph of $ln(x)$ seems to flatten out, it actually will grow without bound, to any arbitrarily large number. To reach an output of B, we need only input e^B, because $\ln(e^B) = B$. That is all to say, the limit in question is ∞.

b. [3 points] The weight w of a round melon is proportional to the cube of its radius r. That is,

$$w = kr^3,$$

where k is a constant. Currently, the melon's radius is 8 cm, and it weighs 5 pounds. How much would it weigh if its radius were to grow to 12 cm? *Give your answer in exact form or rounded to at least two decimals.*

Solution: We can use the information given to solve for the proportionality constant k:

$$5 = k8^3$$
$$k = \frac{5}{8^3}$$

Now we can use that value of k to find the weight w when $k = 12$ cm.

$$w = \frac{5}{8^3} \cdot 12^3 = 16.875$$

Answer: 16.875 pounds