Math 105 — First Midterm — October 8, 2024

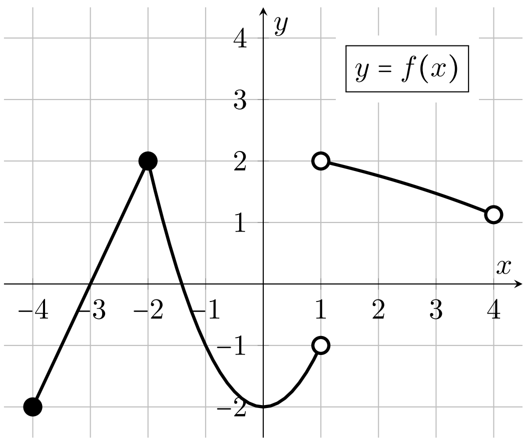
EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	11	
2	11	
3	12	

Problem	Points	Score
4	10	
5	7	
6	9	
Total	60	

1. [11 points] There is information about four different functions below. There is a graph of a function $f(x)$, a piecewise formula for a function $g(x)$, a formula for a function $h(x)$, and a table of some values of an ***invertible*** function $k(x)$.



$$g(x) = \begin{cases} -x & x \le 0 \\ (x-2)^2 & x > 0 \end{cases}$$

$$h(x) = -2(x-1) + 4$$

x	-2	0	4	6
$k(x)$	4	1	8	12

a. [2 points] Let $m(x) = f(x) + 3$. Find the range of $m(x)$. You can express your answer using either intervals or inequalities.

Range: $[1, 5]$

b. [2 points] On which of the intervals below is $f(x)$ **decreasing** on the **entire** interval? Circle **all** correct answers.

$(-3, -1)$ $\boxed{(-2, 0)}$ $(0, 2)$ $\boxed{(1, 3)}$ NONE

c. [2 points] Let $b(x)$ be the linear function which is perpendicular to $h(x)$ and which goes through the point $(8, 3)$. Find a formula for $b(x)$.

$b(x) =$ $\frac{1}{2}(x-8)+3$

d. [5 points] Find or estimate the value of each of the following; write N/A if a value does not exist or there is not enough information to find it.

(i) If $w(x) = f(x-3)$, $w(1) =$ $f(-2) = 2$

(ii) $k^{-1}(4) =$ -2

(iii) $k(h(3)) =$ $k(-2(3-1)+4) = k(-4+4) = k(0) = 1$

(iv) The average rate of change of $g(x)$ from $x = -2$ to $x = 3$.

$\frac{g(3)-g(-2)}{3-(-2)} = \frac{((3-2)^2)-(-(-2))}{5} = \frac{1^2-2}{5} = -\frac{1}{5}$

2. [11 points] The population of Detroit in 1970 was 1.51 million and by 1990 it was 1.03 million. In parts (a)–(c) below, use the following variable definitions:

- Let D be the population of Detroit in **millions of people**.
- Let t be **years since 1970**.

a. [3 points] If we assume the population of Detroit decreased **exponentially** as a function of time, find a formula for $D = E(t)$.

Show all work. Leave constants in exact form, or rounded to at least two decimal places.

Solution: The fact that the population was 1.51 million people in 1970 tells us that $D(0) = 1.51$. Similarly, we have $D(20) = 1.03$ since the population was 1.03 million people in 1990, 20 years after 1970.
If we assume $D = E(t)$ is an exponential function, then it can be written in the form $E(t) = ab^t$ for some constants a and b. We need to solve for a and b. Using the information above, we have

$$E(0) = ab^0 = a = 1.51$$

and

$$E(20) = ab^{20} = 1.03$$

The first equation gives us that $a = 1.51$. Substituting this value of a into the second equation gives us

$$\begin{aligned} 1.51b^{20} &= 1.03 \\ b^{20} &= \frac{1.03}{1.51} \\ b &= \pm \sqrt[20]{\frac{1.03}{1.51}} \end{aligned}$$

Since the growth factor of an exponential function must be positive, we have $b = \sqrt[20]{\frac{1.03}{1.51}} \approx 0.9811$. Therefore, $E(t) = 1.51\left(\sqrt[20]{\frac{1.03}{1.51}}\right)^t$

$D = E(t) =$ $1.51\left(\sqrt[20]{\frac{1.03}{1.51}}\right)^t$

b. [2 points] Is the graph of $D = E(t)$ concave up or concave down or neither? Provide a small sketch or explain *briefly* how you know.

(Circle One) $\boxed{\text{CONCAVE UP}}$ CONCAVE DOWN

Explanation or Sketch:

Solution: An exponential function of the form $P(t) = ab^t$ where $a > 0$ is always concave up.

c. [3 points] If we assume, instead, that the population of Detroit decreased **linearly** as a function of time, find a formula for $D = L(t)$.

Show all work. Leave constants in exact form, or rounded to at least two decimal places.

Solution: The graph of $L(t)$ should go through the points $(0, 1.51)$ and $(20, 1.03)$ as we saw in part (a). The slope of such a linear function is $\frac{1.03-1.51}{20-0} = \frac{-0.48}{20} = -0.024$. Since we already have the value of $L(0)$ is 1.51, we know the vertical intercept is 1.51. Therefore $L(t) = -0.024t + 1.51$. Alternatively, we could use the point $(20, 1.03)$ and write the answer using point-slope form: $L(t) = -0.024(t-20) + 1.03$.

$D = L(t) =$ $\underline{-0.024(t-20) + 1.03 = -0.024t + 1.51}$

d. [3 points] Fort Myers, Florida had a population of 105,260 at the beginning of 2022, which grew by 6.82% over the next year. If the population continues to grow exponentially, how large will the city be in 2030?

Show all work. Leave answer in exact form, or rounded to the nearest whole number.

Solution: Let $F(t)$ be the population of Fort Myers, Florida t years since the start of 2022. If we assume the population is growing exponentially, then $F(t)$ has the form $F(t) = ab^t$. The fact that the population grew by 6.82% over one year (note that this agrees with the units of t) tells us that $r = 0.0682$, so $b = 1.0682$. Since we know $F(0) = 105,260$, we also have that the initial quantity is $a = 105,260$.
More explicitly, we could also solve for these values using the formula for $F(t)$: We are given that $F(0) = 105,260$ and $F(1) = 1.0682(F(0))$.
Therefore,

$$F(0) = ab^0 = a = 105,260$$

and

$$F(1) = ab = 1.0682(105,260)$$

The first equation tells us that $a = 105,260$. Substituting this into the second equation gives us that

$$105,260b = 1.0682(105,260)$$
$$b = 1.0682$$

So, $F(t) = 105,260(1.0682)^t$. The year 2030 corresponds to $t = 8$ since 2030 is 8 years after 2022. We can compute $F(8) = 105,260(1.0682)^8 \approx 178437$ to see that the population would be $105,260(1.0682)^8 \approx 178,437$ people in 2030.

Population in 2030: $\underline{105,260(1.0682)^8 \approx 178,437}$ **people**

3. [12 points]

- The amount of money Bizzi earns in one week, W (in US Dollars), is a function of the number of hours she works that week, h. That is, $W = f(h)$.
- On the other hand, the number of hours she can spend watching K-pop videos, K, is *also* a function of the number of hours she works that week, h. That is, $K = g(h)$.

Both $W = f(h)$ and $K = g(h)$ are invertible functions.

a. [6 points] Describe the meaning of each of the following expressions or equations in the context of Bizzi's life, or explain why the expression or equation doesn't make sense in context.

(i) $f^{-1}(1210) = 50$

Solution: Bizzi must work 50 hours in a week to earn \$1210 that week.
To see this, note that the function f takes as input the number of hours Bizzi works in a week and outputs the amount of money she earns that week, in dollars. Therefore, the inverse function f^{-1} should take as input the amount Bizzi earned, in dollars, in one week and output the number of hours she worked that week.

(ii) $g(f(30)) = 2$

Solution: This composition does not make sense in the context of the problem. The output of f is an amount of money that Bizzi earns in a week. Therefore, $f(30)$ represents some amount of money, in dollars. However, g takes an input a number of hours, not an amount of money in dollars. Therefore, it does not make sense to plug $f(30)$ into g in this context.

(iii) $g(40)$

Solution: This represents the number of hours Bizzi can spend watching K-pop videos if she works 40 hours in a week.

b. [6 points] For each of the following phrases or sentences, write an expression or equation to represent it symbolically using the functions f, g, or their inverses.

(i) Bizzi does *not* work on January 1, but gets paid that week as if she worked 8 hours that day. Express the amount of money Bizzi earns the week of January 1 if she works h hours total during the remainder of the week.

Solution: $f(h+8)$. Bizzi gets paid for the h hours she actually worked plus an additional 8 hours.

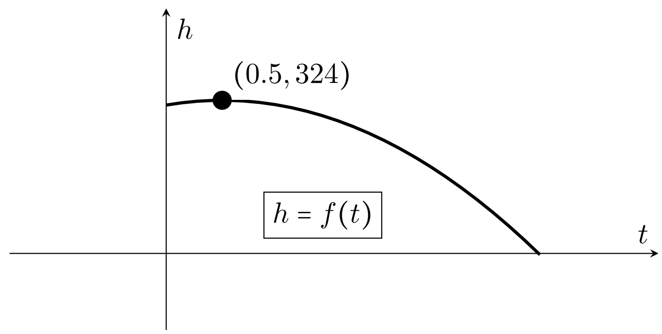
(ii) The week of April 1, every employee in the company gets a \$200 bonus. Express the amount of money Bizzi earns the week of April 1 if she works h hours.

Solution: $f(h) + 200$. Bizzi gets paid an additional \$200 on top of the $f(h)$ dollars she would normally get paid for working h hours.

(iii) Represent the following sentence as an equation: On a week where Bizzi earns 950 dollars, she has time to watch 3 hours of K-Pop videos.

Solution: $g(f^{-1}(950)) = 3$ OR $f(g^{-1}(3)) = 950$.
Looking at units, we see that 950 must be an output of f or input of f^{-1} and 3 must be an output of g or input of g^{-1}. To actually translate this into an equation, we note that if Bizzi earns 950 dollars in a week, she must have worked $f^{-1}(950)$ hours. This would give her $g(f^{-1}(950))$ hours to watch K-pop videos, so $g(f^{-1}(950)) = 3$. Alternatively, if Bizzi had 3 hours to watch K-pop videos in a week, she must have worked $g^{-1}(3)$ hours. This would mean she earned $f(g^{-1}(3))$ dollars, so $f(g^{-1}(3)) = 950$.

4. [10 points] The AT&T Building in Detroit (photo below, left) is 320 feet tall. A ball is (safely) launched from the level of the roof of the AT&T Building. Its height above the ground h, in feet, is a quadratic function of time t, in seconds. A graph of $h = f(t)$ is shown below, to the right. The parabola's vertex is the point $(0.5, 324)$.



a. [2 points] Given the context of this problem, what is the vertical intercept of the graph $h = f(t)$? Explain how you know.

Vertical intercept: **320**

Explanation:

Solution: The AT&T building is 320 feet tall, and the ball is thrown from the top of the building at $t = 0$.

b. [2 points] How many seconds after launch did the ball reach its maximum height and how high was that?

Seconds after launch when maximum height was reached: **0.5 seconds**

Maximum height reached: **324** feet

c. [3 points] Use the information from parts (a) and (b) to find a formula for $h = f(t)$. *Show all work.*

Solution: The function $h = f(t)$ is a quadratic function whose vertex is $(0.5, 324)$. Therefore, when written in vertex form its formula looks like $h = f(t) = a(t-0.5)^2 + 324$ for some constant a. To solve for a, we can use the fact that $f(0) = a(0-0.5)^2 + 324 = 320$. Therefore

$$\begin{aligned} a(-0.5)^2 + 324 &= 320 \\ 0.25a + 324 &= 320 \\ 0.25a &= -4 \\ a &= -16 \end{aligned}$$

so $f(t) = -16(t-0.5)^2 + 324$.

$f(t) =$ $-16(t-0.5)^2 + 324$

This problem continues on the next page.

d. [3 points] A ball is thrown down from a hovercraft cruising above *Jupiter*. The ball's height above Jupiter's surface, h, in feet, is given by:

$$h = -40t^2 - 20t + 560$$

where t is measured in seconds after the ball was released.

From the moment the ball was released, how many seconds did it take for the ball to reach the surface of Jupiter? *Show all work. Give your answer in exact form, or rounded to at least two decimal places.*

Solution: The ball is on the surface of Jupiter when its height h above the surface of Jupiter is 0 feet. Therefore, we are looking for a positive value of t such that $-40t^2 - 20t + 560$ is equal to 0. We can solve for the values of t which make this expression 0 by factoring it:

$$\begin{aligned} -40t^2 - 20t + 560 &= 0 \\ -20(2t + t - 28) &= 0 \\ 2t + t - 28 &= 0 \\ (2t-7)(t+4) &= 0 \\ t = \frac{7}{2} &\text{ or } t = -4 \end{aligned}$$

We don't consider $t = -4$ since it would correspond to a time before the ball was thrown, so we see that it took $t = 7/2$ seconds for the ball to reach the surface of Jupiter.
We could also have computed this answer using the quadratic formula with $a = -40$, $b = -20$, and $c = 560$. This would give us

$$\begin{aligned} t &= \frac{-(-20) \pm \sqrt{(-20)^2 - 4(-40)(560)}}{2(-40)} \\ &= \frac{20 \pm \sqrt{400 + 160(560)}}{-80} \\ &= \frac{20 \pm \sqrt{90,000}}{-80} \\ &= \frac{20 \pm 300}{-80} \end{aligned}$$

as the solutions to the equation $-40t^2 - 20t + 560 = 0$. These two solutions simplify to $\frac{20+300}{-80} = \frac{320}{-80} = -4$ and $\frac{20-300}{-80} = \frac{-280}{-80} = \frac{7}{2}$. Taking the positive value of t, $\frac{7}{2}$, gives us the same solution as factoring did above.

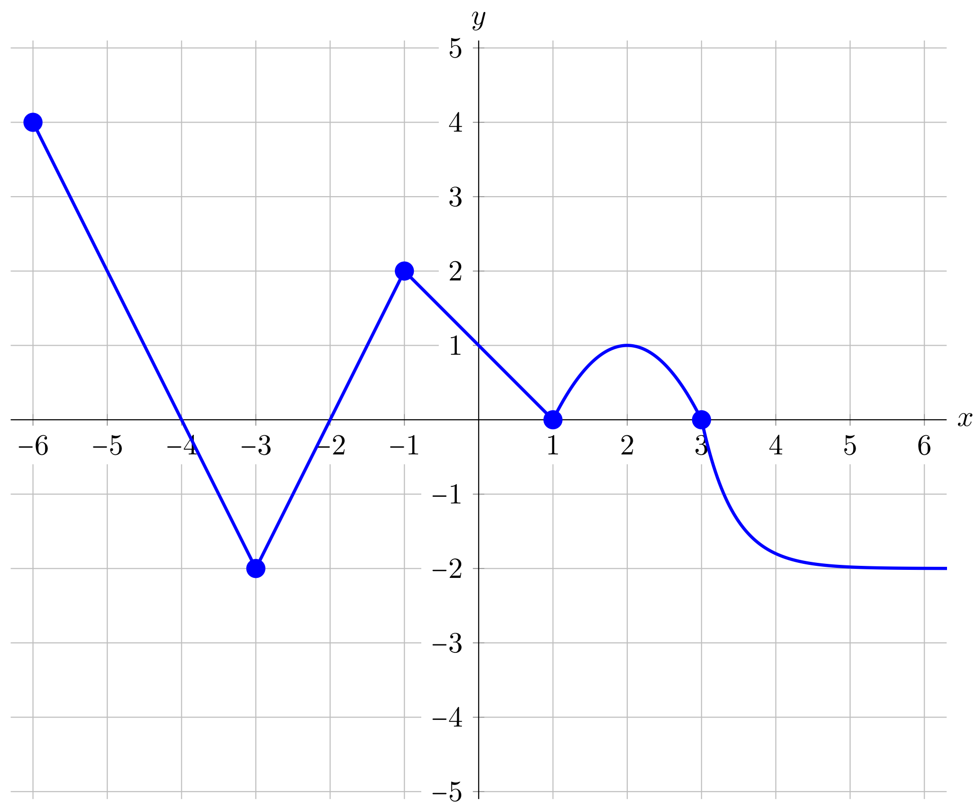
7/2=3.5 seconds

5. [7 points] On the axes below, sketch a graph of a **single function** $j(x)$ that satisfies all of the following properties:

- $j(x)$ has zeros at $x = 1$ and $x = 3$.
- The domain of $j(x)$ is $-6 \le x < \infty$.
- $j(x) \longrightarrow -2$ as $x \longrightarrow \infty$. In other notation: $\lim_{x\to\infty} j(x) = -2$.

- $j(x)$ is decreasing on the interval $[-6, -4]$.
- The average rate of change of $j(x)$ on the interval $-3 \le x \le -1$ is 2.
- $j(x)$ is concave down on the interval $1 < x < 3$.

Solution: There are many possible solutions. Below is just one possibility.



6. [9 points] Isabel's friend Kei lives in the next town over. The two friends are curious about how their water bills compare. Let $I(w)$ be the amount, in dollars, Isabel pays for her water bill for a month if she uses w Centum Cubic Feet (CCFs) of water that month. Let $K(w)$ be the amount, in dollars, Kei pays for their water bill for a month if they use w CCFs of water that month. Both functions are linear and their formulas are:

$$I(w) = 4.1w + 25 \qquad K(w) = 4.5w + 15$$

a. [3 points] Find $K^{-1}(33)$ and write a sentence which explains what the value you find means in the context of the problem. *Show all work. Give your answer in exact form or rounded to at least two decimal places.*

Solution: We can find $K^{-1}(33)$ by solving $K(w) = 4.5w + 15 = 33$ for w. Subtracting 15 from both sides gives us $4.5w = 18$. If we divide by 4.5, we find $w = 18/4.5 = 4$.

$K^{-1}(33) =$ **4**

Meaning:

Solution: If Kei's water bill for a month is \$33, they must have used 4 CCFs of water that month.

b. [1 point] If Kei used two more CCFs of water in August than in June, how much more expensive was their August water bill than their June water bill? *You do not need to show any work.*

Kei's August water bill is **9** dollars more than their June water bill.

Solution: Let c be the amount of water, in CCFs, that Kei used in June. This means they used $c + 2$ CCFs of water in August. Then their bill in June is $K(c) = 4.5c + 15$, and their bill in August is $K(c + 2) = 4.5(c + 2) + 15$. The difference is

$$\begin{aligned} K(c+2) - K(c) &= 4.5(c+2) + 15 - (4.5c + 15) \\ &= 4.5c + 4.5(2) + 15 - 4.5c - 15 \\ &= 4.5(2) \\ &= 9 \end{aligned}$$

. Therefore, their August bill is \$9 more than their June water bill.

c. [2 points] What is the amount of water usage (in CCFs) that would cost the same amount under both water bill plans? *Show all work. Give your answer in exact form, or rounded to at least two decimal places.*

Solution: We want to find the amount of water w which makes $I(w) = K(w)$. Therefore, we need to solve $4.1w + 25 = 4.5w + 15$ for w:

$$\begin{aligned} 4.1w + 25 &= 4.5w + 15 \\ 25 &= 0.4w + 15 \\ 10 &= 0.4w \\ w &= \frac{10}{0.4} \\ w &= \frac{100}{4} \\ w &= 25 \end{aligned}$$

25 CCFs

Let $g(t)$ be the number of CCFs of water Kei's household has used t days since the start of June (so $t = 1$ would correspond to 12:00am on June 2nd). Some values of $g(t)$ are displayed in the table below.

t	1	5	7	11
$g(t)$	5	19.5	28	33

d. [3 points] Kei's family went out of town (and therefore didn't use any water at home) for a couple days during June. Based on the table above, during which of the following time periods is most likely that Kei's family went out of town?

Circle the **one** best possible answer. *Show all work and explain why you circled the option you chose.*

June 3rd to June 5th June 6th to June 8th $\boxed{\text{June 9th to June 11th}}$

Explanation:

Solution: We can compute the average rate of change of $g(t)$ on the intervals given in the table to estimate how much water Kei's family used per day, on average, during each of those intervals.

On the first interval from $t = 1$ to $t = 5$, the average rate of change is $\frac{g(5)-g(1)}{5-1} = \frac{19.5-5}{5-1} = \frac{14.5}{4} = 3.625$ CCFs/day. That means that between 12am on June 2nd and 12am on June 6th, Kei's household used an average of 3.625 CCFs of water per day.

On the interval from $t = 5$ to $t = 7$, the average rate of change is $\frac{g(7)-g(5)}{7-5} = \frac{28-19.5}{7-5} = \frac{8.5}{2} = 4.25$ CCFs/day. So, from 12am on June 6th to 12am on June 8th, Kei's household used an average of 4.25 CCFs of water per day.

Finally, on the interval from $t = 7$ to $t = 11$, the average rate of change is $\frac{g(11)-g(7)}{11-7} = \frac{33-28}{11-7} = \frac{5}{4} = 1.25$ CCFs/day. Therefore, from 12am on June 8th to 12am on June 12th, Kei's household used on average 1.25 CCFs of water per day.

We see that Kei's household had the lowest average water usage per day between June 8th and June 12th. Therefore, the most likely option for when their family went out of town is June 9th to June 11th.