Math 105 — Second Midterm — November 12, 2024

EXAM SOLUTIONS

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 11 pages including this cover. There are 7 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is <u>not</u> permitted.

Problem	Points	Score
1	13	
2	10	
3	6	

Problem	Points	Score
4	6	
5	11	
6	6	
7	8	
Total	60	

1. [13 points] A formula for the function $K(x)$ and a table of values for the **odd** function $A(x)$ are shown below. **The domain of $A(x)$ is all real numbers.**

$$K(x) = x^2 - 6$$

x	1	5	10
$A(x)$	-2	-6	-10

a. [5 points] If possible, evaluate each of the following expressions. If the value does not exist, write DNE; if there is not enough information to determine it, write NEI.
No explanations needed; work shown may earn partial credit.

- If $D(x) = A(x) + K(x)$, then $D(10) =$ $A(10) + K(10) = -10 + 94 = 84$

- $A(K(1)) =$ $A(-5) = 6$ **because we know that $A(x)$ is odd.**

- $K^{-1}(-7) =$ **DNE**

Solution: This is because -7 is not in the range of $K(x)$ and *also* because $K(x)$ is not invertible.

- $A(0) =$ **0**

Solution: We know that $A(0)$ is defined and, further:

$$A(0) = -A(-0) = -A(0)$$

because $A(x)$ is odd. But this also shows that $A(0)$ is its own negative, so must be 0.

b. [8 points] For each of the following functions, decide if it is *odd*, *even*, or *neither*. Circle one answer for each part. *Show all work for full credit.*

(i) $K(x)$

ODD   [EVEN]   NEITHER

Solution: We want to see if $K(-x) = K(x)$ (even), $K(-x) = -K(x)$ (odd), or neither.

$$K(-x) = (-x)^2 - 6 = x^2 - 6 = K(x),$$

so $K(x)$ is even.

Alternatively, we could see that the graph of $K(x)$ is a parabola with vertex at $(0, 0)$ that's been reflected over the x-axis and then shifted down 6. So it's still symmetrical about the y-axis.

(ii) $K(x) + A(x)$

ODD   EVEN   [NEITHER]

Solution: We want to see if $K(-x)+A(-x)=K(x)+A(x)$ (even), $K(-x)+A(-x)=-K(x)-A(x)$(odd), or neither.
Let's test two points $x=\pm 1$:

$$K(-1)+A(-1)=-5+2=3$$

$$K(1)+A(1)=-5-2=-7$$

Our answers are neither the same, nor opposites, so the function is neither even nor odd.

(iii) $A(K(x))$

ODD   [EVEN]   NEITHER

Solution: Since $K(-x)=K(x)$, we have $A(K(-x))=A(K(x))$. This means $A(K(x))$ is even.

(iv) $A(x)\cdot K(x)$

[ODD]   EVEN   NEITHER

Solution: Because $A(x)$ is odd we know $A(-x)=-A(x)$; and because $K(x)$ is even we know $K(-x)=K(x)$. Therefore,

$$A(-x)\cdot K(-x)=-A(x)\cdot K(x)=-(A(x)\cdot K(x)),$$

showing that $A(K(x))$ is odd.

2. [10 points] A tank is full of a fixed amount of neon gas.

- The function $f(T)$ gives the pressure exerted by the neon gas on the tank, in Pascals (Pa), when the gas in the tank is T °C.
- The function $g(t)$ gives the temperature of the gas in the tank, in °C, t minutes after the heating source was turned on.

a. [4 points] Give an expression, involving the functions g, f or their inverses, that represents each of the following quantities.

(i) The pressure of the gas in the tank, in Pa, 4 minutes after the heating source was turned on.

Answer: $f(g(4))$

(ii) The temperature of the neon gas, in °C, h **hours** after the heating source was turned on.

Answer: $g(60h)$

b. [6 points] For each of the following expressions or equations, write a phrase or sentence describing what it means in the context of the problem. If the expression or equation does not make sense in the context of the problem, write "NA" and explain why not. **Make sure to include all relevant units.**

(i) $f^{-1}(50) = 60$ **Meaning/Explanation:**

Solution: When the gas exerts 50Pa of pressure, the temperature of the gas in the tank is 60°C.

(ii) $f^{-1}(g^{-1}(12))$ **Meaning/Explanation:**

Solution: This expression does not make sense in context. The quantity $g^{-1}(12)$ represents a time in minutes, but f^{-1} takes as input pressure in Pascals.

(iii) $g(0) = g(5) + 10$ **Meaning/Explanation:**

Solution: Possible solutions:

- The temperature decreases by 10° in the first five minutes.
- Or, if we assume the heating sources adds heat (and not subtracts it, such as with a heat pump), then we could also say that this doesn't make sense in the context of the problem.

3. [6 points] Returning to the scenario in the previous problem: a tank is full of a fixed amount of neon gas and the function $g(t)$ gives the temperature of the gas in the tank T (in °C) as a function of t, where t is measured in minutes since a heating source was turned on. Now we learn further that, for some constant $c < 0$:

$$g(t) = 300 - 250e^{ct}$$

a. [3 points] Assume that the domain for $g(t)$ in this context is $[0, \infty)$. In that case, what is the associated *range* of $g(t)$ and what does this mean in the context of the problem? *Show all relevant work.*

Range: $[50, 300)$

Solution: Note that $g(t)$ is an increasing function since $250e^{ct}$ is a decreasing function if $c < 0$. We have that $g(0) = 300 - 250e^0 = 50$. On the other hand, since $250e^{ct}$ is always positive but approaches 0 as $t \to \infty$, $g(t)$ is always less than 300 but approaches 300 as $t \to \infty$.

Meaning:

Solution: The gas starts at a temperature of 50°C when the heating source is turned on and approaches a temperature of 300°C as time goes on.

b. [3 points] If 30 minutes after the heating source is turned on the temperature of the gas in the tank reaches 200°C, what must be the value of c? *Show all work. Leave your answer in exact form.*

$c = \frac{\ln(2/5)}{30}$

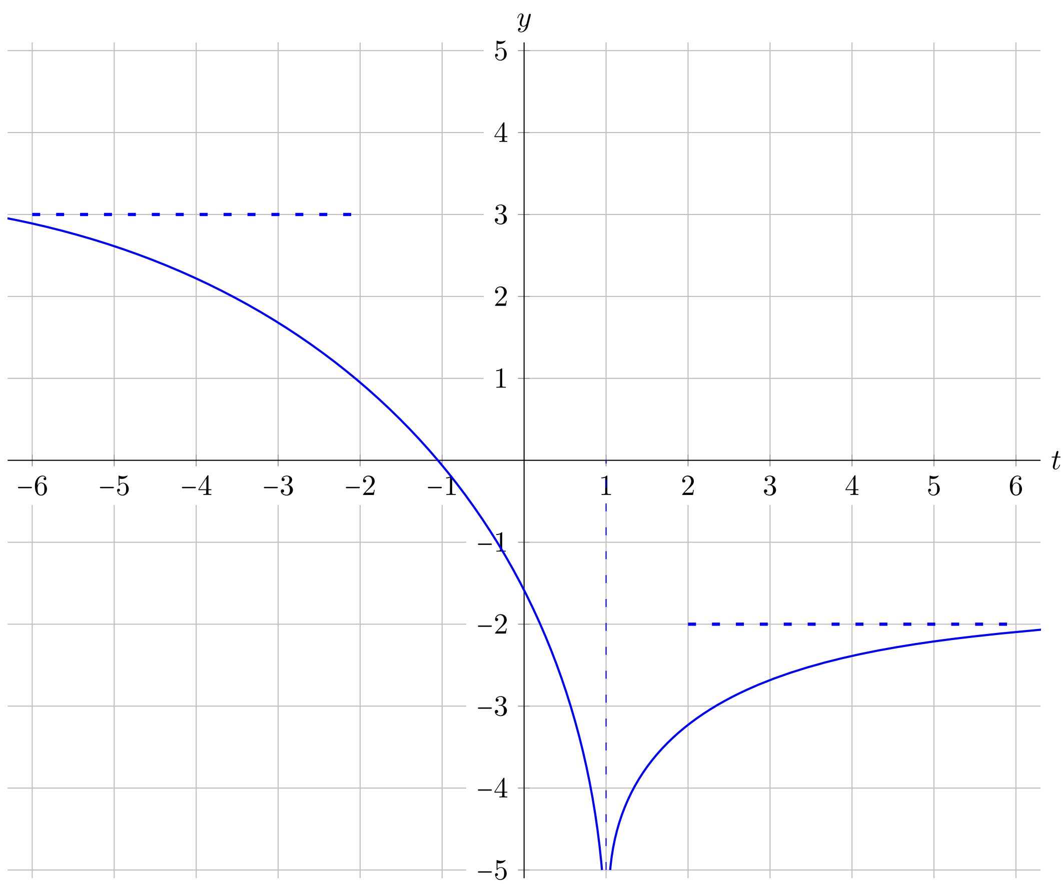
Solution: We are given that $g(30) = 200$. That is, $300 - 250e^{30c} = 200$. We need to solve this equation for c:

$$\begin{aligned} 300 - 250e^{30c} &= 200 \\ -250e^{30c} &= -100 \\ e^{30c} &= \frac{-100}{-250} \\ e^{30c} &= \frac{2}{5} \\ 30c &= \ln(2/5) \\ c &= \frac{\ln(2/5)}{30} \end{aligned}$$

4. [6 points] On the axes below, sketch the graph of a single function $J(t)$ that satisfies all of the properties below. Clearly indicate any relevant features of your graph.

- The domain of $J(t)$ is all real numbers except for $t = 1$.
- The range of $J(t)$ is $(-\infty, 3)$
- $\lim_{t\to\infty} J(t) = -2$.
- $\lim_{t\to-\infty} J(t) = 3$.
- $J(t)$ has a vertical asymptote at $t = 1$.

Solution: There are several possible graphs; one possibility that fits all the criteria is shown below.



5. [11 points] Jasen has decided to learn another language. The number of months y that it takes Jasen to learn x thousands of words of the language is given by

$$y = f(x) = -50\ln\left(\frac{1}{19}(-x+20)\right).$$

a. [8 points] List the transformations you need to apply (in order!) to the graph of $y = \ln(x)$ to transform it to that of $y = f(x)$ above. Fill in the first blank with one of the phrases below. In the second blank, write a number that represents the appropriate shift or scaling factor; leave it blank if the first blank was a reflection. In the third blank, write an equation for the vertical asymptote of the intermediate graph.

SHIFT IT TO THE LEFT   STRETCH IT HORIZONTALLY   REFLECT IT ACROSS THE y-AXIS

SHIFT IT TO THE RIGHT   COMPRESS IT HORIZONTALLY   REFLECT IT ACROSS THE x-AXIS

SHIFT IT UP   SHIFT IT DOWN   STRETCH IT VERTICALLY   COMPRESS IT VERTICALLY

We start with $\ln(x)$ (vertical asymptote at $x = 0$).

First, **Stretch it vertically** by **50** (vertical asymptote at $x =$ **0**).

Then, **reflect it across the x-axis** by _____ (vertical asymptote at $x =$ **0**).

Then, **stretch it horizontally** by **19** (vertical asymptote at $x =$ **0**).

Then, **shift it to the left** by **20** (vertical asymptote at $x =$ **-20**).

Then, **reflect it across the y-axis** by _____ (vertical asymptote at $x =$ **20**).

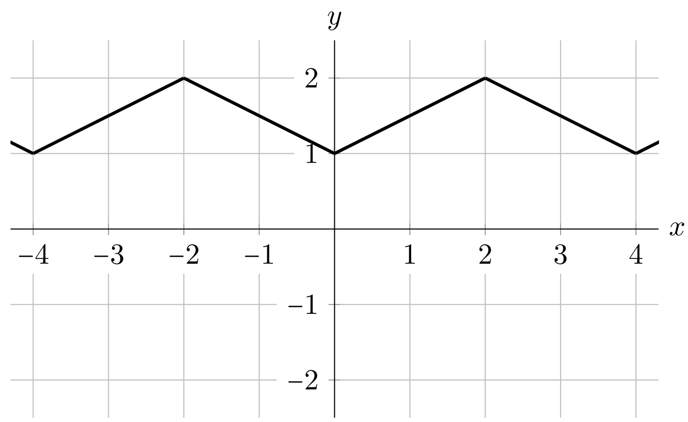
b. [3 points] Find a formula for $x = f^{-1}(y)$. *Show all work.*

Solution: We solve the equation $y = -50\ln\left(\frac{1}{19}(-x+20)\right)$ for x in terms of y.

$$\begin{aligned}
y &= -50\ln\left(\frac{1}{19}(-x+20)\right) \\
-y &= 50\ln\left(\frac{1}{19}(-x+20)\right) \\
-\frac{y}{50} &= \ln\left(\frac{1}{19}(-x+20)\right) \\
e^{-y/50} &= \frac{1}{19}(-x+20) \\
19e^{-y/50} &= -x+20 \\
19e^{-y/50} - 20 &= -x \\
x &= 20 - 19e^{-y/50}
\end{aligned}$$

$x = f^{-1}(y) =$ $20 - 19e^{-y/50}$

6. [6 points] The function $h(x)$ is a periodic function which is defined for all real numbers. The period of $h(x)$ is 4. A portion of the graph of $h(x)$ is shown on the graph below.

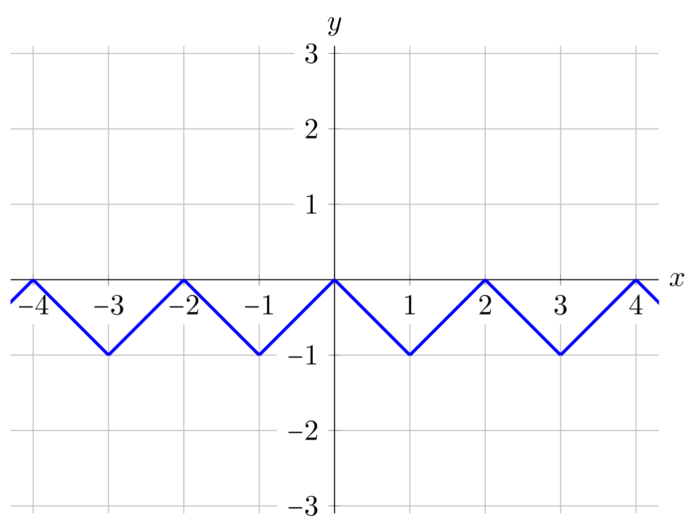


a. [3 points] Let $j(x)$ be the function whose graph is obtained by performing the following transformations to the graph of $h(x)$, in the following order:

1. Horizontal compression by a factor of 1/2
2. Vertical reflection across the x-axis
3. A vertical shift up by 1

On the axes below, sketch a graph of $j(x)$.

Feel free to use the upper set of axes for sketching intermediate steps and make sure your final graph is clearly indicated on the axes below.



b. [3 points] The function $j(x)$ is also periodic. What are the period, amplitude and midline of $j(x)$?

Period: 2 **Amplitude:** 0.5 **Midline:** $y =$ −0.5

7. [8 points] For each question below, circle TRUE or FALSE or NEI (Not Enough Information). Justify your answer *briefly. No partial credit if explanation is left blank.*

a. [2 points] The exponential function $f(t) = e^{1.02t}$ grows more slowly than the exponential function $g(t) = 1.05^t$.

TRUE **FALSE** NEI

Explanation:

Solution: The growth factor of $f(t) = e^{1.02t} = (e^{1.02})^t$ is $e^{1.02} > e \approx 2.7$ while the growth factor of $g(t)$ is 1.05.

b. [2 points] If a exponential function $E(t)$ decays by 15% between $t = 0$ and $t = 2$, it will also decay by 15% between $t = 1$ and $t = 3$.

TRUE FALSE NEI

Explanation:

Solution: Exponential functions have a constant percent decay rate. Therefore, $E(t)$ will decay by the same percent over these two intervals of the same width.

c. [2 points] The function $\log(x)$ grows *so* slowly that it eventually approaches a horizontal asymptote.

TRUE **FALSE** NEI

Explanation:

Solution: The limit as $x \to \infty$ of $\log(x)$ is ∞. In fact, for any value a you'd want to see as an output of $\log(x)$, you can input 10^a and you'll get $\log(10^a) = a$.

d. [2 points] If the doubling time of exponential function $h(t)$ is 25 years, then its annual growth factor is $\frac{\log 2}{\log 25}$.

TRUE **FALSE** NEI

Explanation:

Solution: Since $h(t)$ is exponential, it has the form $h(t) = ab^t$. If the doubling time is 25 years, then $ab^{25} = h(25) = 2h(0) = 2ab^0 = 2a$. Therefore, $b^{25} = 2$, and the growth factor is $b = 2^{1/25} \approx 1.03$ while $\frac{\log 2}{\log 25} \approx 0.22$.