Math 105 — Final Exam — December 12, 2024

EXAM SOLUTIONS

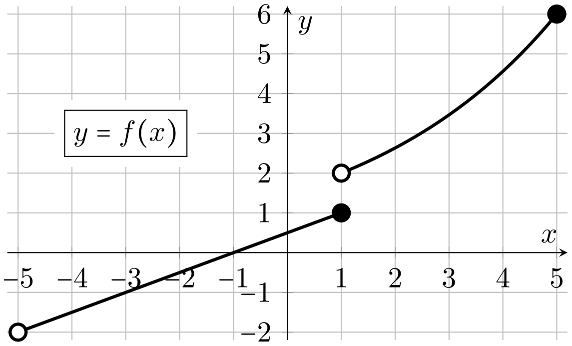
1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 18 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID (not name) on every page and point this out to your instructor when you hand in the exam.
5. The back of every page of the exam is blank, and, if needed, you may use this space for scratch-work. Clearly identify any of this work that you would like to have graded.
6. Read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so while you may ask for clarification if needed, instructors are generally unable to answer such questions during the exam.
7. Show an appropriate amount of work for each problem, so that graders can see not only your answer but how you obtained it.
8. You must use the methods learned in this course to solve all problems.
9. You are allowed notes written on two sides of a $3'' \times 5''$ note card and one scientific calculator that does not have graphing or internet capabilities.
10. Include units in your answer where that is appropriate.
11. Problems may ask for answers in *exact form* or in *decimal form*. Recall that $\sqrt{2} + \cos(3)$ is in exact form and 0.424 would be the same answer expressed in decimal form.
12. **Turn off all cell phones, smartphones, and other electronic devices**, and remove all headphones, earbuds, and smartwatches. Put all of these items away. The use of any networked device while working on this exam is not permitted.

Problem	Points	Score
1	14	
2	14	
3	6	
4	6	
5	6	

Problem	Points	Score
6	9	
7	7	
8	11	
9	7	
Total	80	

1. [14 points]

The entire graph of a function $f(x)$ is shown to the right. Also shown is a table of some values for two functions $g(x)$ and $h(x)$.



x	0	1	2	4
$g(x)$	2	−1	5	4
$h(x)$	0	0.5	0	8

The function $g(x)$ is defined for all real numbers and is **periodic** with a period of 5.

a. [3 points] Find the value of each of the following; write N/A if a value does not exist or there is not enough information to find it. *Showing work is not required, but may be eligible for partial credit in some cases.*

(i) $h(f(1)) =$ $h(1) = 0.5$

(ii) $g(\sin(20\pi)) =$ $g(0) = 2$

(iii) $g(f^{-1}(-1)) =$ $f(-3) = f(2) = 5$

b. [3 points] Suppose that we know further that $h(x)$ is a polynomial of **degree 3** with a double zero at $x = 2$. Combining this new knowledge with what's given in the table, find a formula for $h(x)$. *Show all work.*

Solution: Given that information about the zeros, we know that $h(x)$ must be of the form:

$$h(x) = ax(x-2)^2$$

for some constant a. To find the value of a we can plug in some known input/output combination and solve.

$$8 = a(4)(4-2)^2$$

$$8 = a(16)$$

$$\frac{1}{2} = a$$

$h(x) =$ $\frac{1}{2}x(x-2)^2$

c. [6 points] The piecewise function $f(x)$ consists of a linear piece and an exponential piece. Write a piecewise formula for the function $f(x)$. *Show all needed work.*

Solution: The linear part of the graph (on the left) has a domain of $-5 < x \leq 1$. We can see that the line goes up 1 for every 2 to the right, so has a slope of $\frac{1}{2}$. Finally, we can use point slope form and the fact that the line goes through $(1, 1)$ to find the formula for the linear part: $y = \frac{1}{2}(x - 1) + 1$.

For the exponential part, we can notice first that the domain is $1 < x \leq 5$. There are a few ways to approach finding the formula. One way is to notice right away that the output multiplies by 3 (from 2 to 6) over 4 steps of x, meaning that $b^4 = 3$ so the growth factor is $b = 3^{\frac{1}{4}}$. If we think about that exponential part as a shift, then we can find a shortcut and think of it as $2(3^{\frac{1}{4}})^x$ shift one to the right, resulting in the formula: $2(3^{\frac{1}{4}})^{x-1}$.
We could also solve this by setting up a system of equations and unknown parameters (a and b) and use two known points.

$$2 = ab^1$$

$$6 = ab^5$$

Approaching it this way leads to an equivalent result:

$$\frac{2}{3^{\frac{1}{4}}}(3^{\frac{1}{4}})^t$$

$$f(x) = \begin{cases} \underline{\frac{1}{2}(x-1)+1} & \text{for } \underline{-5 < x \leq 1} \\ \underline{2(3^{\frac{1}{4}})^{x-1}} & \text{for } \underline{1 < x \leq 5} \end{cases}$$

d. [2 points] Find the domain of the function $f^{-1}(y)$ (*not* $f(x)$).

Domain of $f^{-1}(y)$: $\underline{(-2, 1] \cup (2, 6]}$

Solution: The domain of f^{-1} is the same as the *range* of f.

2. [14 points] Psychiatrists and food scientists teamed up to measure how the concentration of different sweeteners affect the perception of sweetness. “Sweetness units” (SU) in this trial ranged from 1 to 15.

- The function $F(C)$ gives the sweetness units (SU) of a $C\%$ fructose solution. (A concentration of $C\%$ means that, by mass, the fructose was $C\%$ of the total solution and the rest was water.)
- The function $A(C)$ gives the sweetness units (SU) of a $C\%$ Alitame solution. (A concentration of $C\%$ means that, by mass, the Alitame was $C\%$ of the total solution and the rest was water.)

a. [6 points] Write an expression or equation, using A, A^{-1}, F, F^{-1}, or their combinations or compositions, that represents each of the following sentences or phrases.

(i) The perceived sweetness of a 6% fructose solution is 9 SUs.

$F(6) = 9$

(ii) The concentration of Alitame that gives a perceived sweetness of 8 SU

$A^{-1}(8)$

(iii) A 0.01% solution of Alitame is 100 times sweeter than a 1% solution of fructose.

$A(0.01) = 100F(1)$

b. [2 points] Which of the following compositions make sense in the context of the problem? *Bubble in all that apply.*

○ $A(F(15))$

○ $F(A(0.05))$

[✓ ○ $A^{-1}(F(11))$]

[✓ ○ $A(F^{-1}(10))$]

○ $A^{-1}(F^{-1}(5))$

○ NONE OF THE ABOVE

Solution: Both $A(F(15))$ and $F(A(0.05))$ try to plug some number of SUs into a function which takes a concentration as input, which doesn't make sense in the context of the problem. Similarly, $A^{-1}(F^{-1}(5))$ tries to plug in a concentration into a function, A^{-1}, which takes as input a number of SUs.

On the other hand, the output of F and the input of A^{-1} are both a number of SUs, so the composition $A^{-1}(F(11))$ does make sense. It is the concentration an Alitame solution would need to be to have the same perceived sweetness (in SUs) as an 11% fructose solution.

The output of F^{-1} is a concentration, and this is what the function A takes as input. Therefore, $A(F^{-1}(10))$ makes sense in the context of the problem. It is the perceived sweetness (in SUs) of an Alitame solution which has the same concentration as a fructose solution which has a perceived sweetness of 10 SUs.

This information is repeated from the previous page for convenience.

- The function $F(C)$ gives the sweetness units (SU) of a $C\%$ fructose solution. (A concentration of $C\%$ means that, by mass, the Fructose was $C\%$ of the total solution and the rest was water.)
- The function $A(C)$ gives the sweetness units (SU) of a $C\%$ Alitame solution. (A concentration of $C\%$ means that, by mass, the Alitame was $C\%$ of the total solution and the rest was water.)

c. [3 points] The scientists found that $F(C)$ was linear, with a slope of 1.33. Given that, which of the following statements about $F(C)$ are TRUE? *Bubble in all that apply.*

○ A 10% fructose solution will be 33% sweeter than a 9% fructose solution.

○ An 8% fructose solution will be about 4 SUs sweeter than a 5% fructose solution.

○ A 4% fructose solution will be 1.33 SUs *less sweet* than a 5% fructose solution.

○ $F(C)$ is an increasing function.

○ $F(C)$ has a constant average rate of change.

○ NONE OF THE ABOVE

Solution: Since $F(C)$ is linear with a positive slope, it is increasing and has a constant average rate of change (of 1.33). If we increase (resp. decrease) the concentration C by 1, the output, which is a perceived sweetness in SUs, increases (resp. decreases) by 1.33. This means that a a 4% fructose solution will be 1.33 SUs less sweet than a 5% fructose solution. Similarly, an 8% fructose solution will be 3.99 SUs, or about 4 SUs, sweeter than a 5% fructose solution. On the other hand, we have that 10% fructose solution will be 1.33 SUs sweeter than a 9% fructose solution, which is not the same as being 33% sweeter.

d. [3 points] The scientists published the following table of data for the function $A(C)$.

C	0.0025	0.005	0.01	0.02
$A(C)$	6.89	9.36	11.4	12.8

Which of the following statements about $A(C)$ may be TRUE, given data in the table provided. *Bubble in all that apply.*

○ $A(C)$ is increasing.

○ The average rates of change of $A(C)$ are decreasing.

○ $A(C)$ is linear.

○ $A(C)$ is concave up.

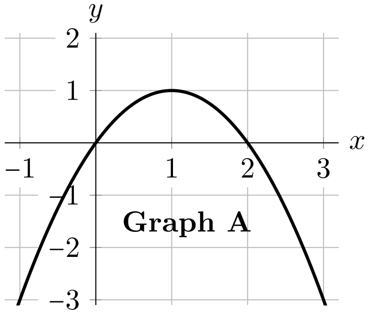
[✓ ○ $A(C)$ is concave down.]

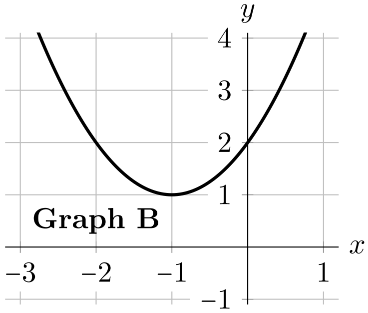
○ NONE OF THE ABOVE

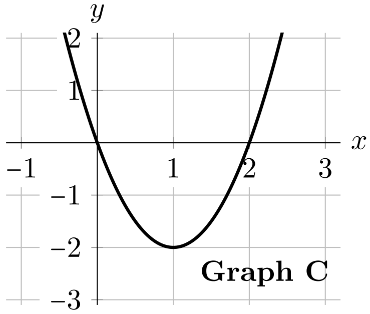
Solution: We can see from the table that $A(C)$ looks like it could be increasing since the values of $A(C)$ increase as the values of C increase (and this makes sense because a more highly concentrated solution should taste sweeter). The average rates of change of $A(C)$ on the intervals 0.0025 to 0.005, 0.005 to 0.01, and 0.01 to 0.02 are 988, 408, and 140, respectively. Since these are not constant, the function is not linear. Since the average rates of change are decreasing, it could be that $A(C)$ is concave down, but it cannot be concave up.

3. [6 points] For each of the following equations, bubble in the letter of the corresponding graph or bubble in **D** if it does not correspond to any of the shown graphs. Use pencil in case you need to change your answer. *A graph may appear as an answer multiple times.*

GRAPHS A, B, C:







(i) $y = -(x-1)^2 + 1$

[✓ ○ A] ○ B ○ C ○ D (None)

Solution: Since the leading coefficient is negative, we see that this function is concave down. It is written in vertex form, so we can read off that it has a vertex at $(1, 1)$. This looks like it lines up with Graph A. To verify, we can plug in $x = 0$ or $x = 2$ to check that they are zeros or rewrite $y = -(x-1)^2 + 1$ in factored form as $-(x^2 - 2x + 1) + 1 = -(x^2 - 2x) = -x(x-2)$.

(ii) $y = x(x-2)$

○ A ○ B ○ C [✓ ○ D (None)]

Solution: Since the leading coefficient is positive, we see that this function is concave up. It is written in factored form, so we can read off that it has zeros at $x = 0, 2$. This sounds like it could match Graph C. To check, we can plug in $x = 1$: this gives a value of $1(1-2) = -1$. However, we see that the vertex of Graph C is at $(1, -2)$ not $(1, -1)$, so this is not a match.

(iii) $y = (x+1)^2 + 1$

○ A [✓ ○ B] ○ C ○ D (None)

Solution: This function is concave up and has a vertex at $(-1, 1)$. It has no zeros, so we can check another point to make sure that Graph B is a match. For example, we have that at $x = 0$, $y = (0+1)^2 + 1 = 2$, which does match Graph B.

(iv) $y = 2(x-1)^2 - 2$

○ A ○ B [✓ ○ C] ○ D (None)

Solution: This function is concave up and has a vertex at $(1, -2)$, so it looks like it could match Graph C. To check, we could plug in $x = 0$ or $x = 2$ to confirm that they are zeros or rewrite the function in factored form as $2(x^2 - 2x + 1) - 2 = 2(x^2 - 2x) = 2x(x-2)$.

(v) $y = x(x+2)+2$

○ A [✓ ○ B] ○ C ○ D (None)

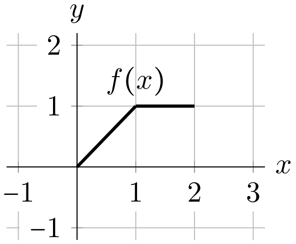
Solution: This function is not written in either factored or vertex form. If we rewrite it in standard form as $x^2 + 2x + 2$, we see it doesn't factor. However, we might see that this is the same as $(x+1)^2+1$ from part (iii) which matched Graph B. Alternatively, we can realize that it is a shift up by 2 of $x(x+2)$. The function $x(x+2)$ is concave up and has zeros at $x = -2, 0$. Therefore, $x(x+2)+2$ should be concave up and go through the points $(-2, 2)$ and $(0, 2)$. This looks like it should match Graph B. To check, we plug in $x = -1$ to get a y-value of $-1(-1+2)+2 = -1+2 = 1$, which does match Graph B.

(vi) $y = 2x(x-2)$

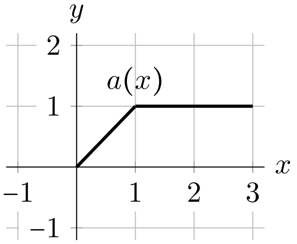
○ A ○ B [✓ ○ C] ○ D (None)

Solution: This function is concave up and has zeros at $x = 0, 2$. This sounds like it could match Graph C. To confirm, we can plug in $x = 1$ to get a y-value of $2(1)(1-2) = 2(-1) = -2$, which does match Graph C.

4. [6 points] The entire graph of $f(x)$ is show below. For each subsequent graph show below, write a formula for the graph shown in terms of $f(x)$. That is, write the formula as a transformation of $f(x)$. Or if it cannot be written as a transformation of $f(x)$, write N/A.

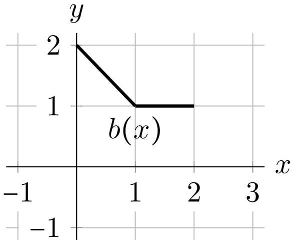


(a)



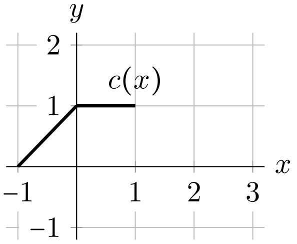
$a(x) =$ **NA**

(b)



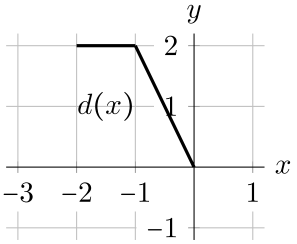
$b(x) = -f(x) + 2$

(c)



$c(x) = f(x+1)$

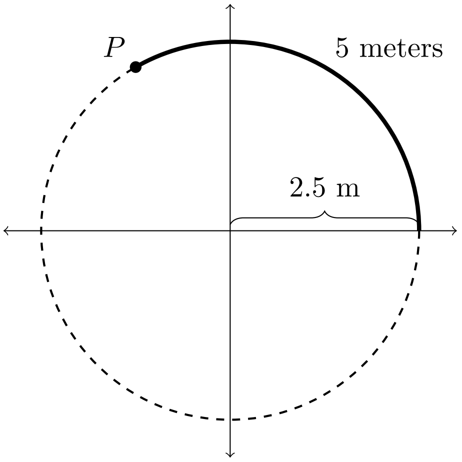
(d)



$d(x) = 2f(-x)$

5. [6 points] The parts of the question below are unrelated.

a. [3 points] The circle below is centered at the origin and has a radius of 2.5 meters. The arc shown in bold has a length of 5 meters. *Give the coordinates of P in exact form or rounded to at least three decimal places.*

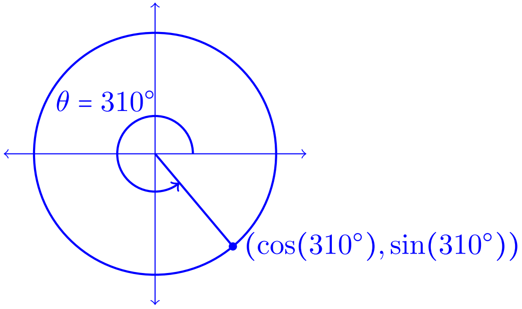


Solution: We know that the arc length spanned in a circle of radius r by an angle of θ radians is given by $r\theta$. In this case, the arc length is 5m and the radius is 2.5m. Therefore, the point P corresponds to an angle of $\theta = 5/2.5 = 2$ radians. We then have that the coordinates of P are $(2.5\cos(2), 2.5\sin(2))$.

Coordinates of P = $(2.5\cos(2), 2.5\sin(2))$

b. [3 points] Trying to find the cosine of 310 *degrees*, Sofia enters $\cos(310)$ into her calculator and gets -0.52534763851. How does she immediately know that something is wrong? *Draw a picture* and explain in words how Sofia knows her calculator must not have been in "degree mode"?

Solution: The cosine of 310 degrees is the x-coordinate of the point on the unit circle which corresponds to an angle of 310 degrees. This point has a positive x-coordinate, as pictured below. Therefore, Sofia was expecting a positive number and knows something is wrong when she gets a negative number.



6. [9 points] Bubble in the blanks for **all possible** correct choices. Use pencil in case you need to change your answer. *You do not need to show work for any part of this problem.*

a. Which of the functions below have the property that: $\lim_{x\to\infty} f(x) = \infty$?

○ $f(x) = \dfrac{9}{x^5}$

○ $f(x) = \dfrac{x^9+5}{2^x+x^2}$

[✓ ○ $f(x) = \dfrac{x^2}{e^{-x}}$]

[✓ ○ $f(x) = \dfrac{x^{\frac{1}{3}}}{\ln(x)}$]

○ NONE OF THE ABOVE

Solution: Let's find the limit as $x \to \infty$ of each of the options.

- Since $x^5 \to \infty$ as $x \to \infty$, $\frac{9}{x^5} \to 0$ as $x \to \infty$.
- We have that $\lim_{x\to\infty} \frac{x^9+5}{2^x+x^2} = \lim_{x\to\infty} \frac{x^9}{2^x} = 0$ since 2^x dominates x^9 as $x \to \infty$.
- As $x \to \infty$, $e^{-x} \to 0$ but is always positive. Since $x^2 \to \infty$ and is always positive as $x \to \infty$, we have, $\lim_{x\to\infty} \frac{x^2}{e^{-x}} = +\infty$.
- Both $x^{1/3}$ and $\ln(x)$ go to ∞ as $x \to \infty$. However, $x^{1/3}$ dominates $\ln(x)$, so we still have $\lim_{x\to\infty} \frac{x^{1/3}}{\ln(x)} = \infty$.

b. Which of the following functions have at least one *horizontal* asymptote?

○ $f(x) = \log(5x)$

[✓ ○ $f(x) = \dfrac{5x^5-x^2}{x^5+x^4}$]

[✓ ○ $f(x) = \dfrac{2}{x^3-x-17}$]

[✓ ○ $f(x) = e^{3x}+1$]

○ $f(x) = \dfrac{x^3+x+71}{x^2-81}$

○ NONE OF THE ABOVE

Solution:

- $\log(5x)$ only has a vertical asymptote and no horizontal asymptotes.
- Since the numerator and denominator of $\frac{5x^5-x^2}{x^5+x^4}$ are polynomials of the same degree, the function has a horizontal asymptote of 5 (which is the ratio of the leading coefficients of the numerator and denominator).
- $\lim_{x\to\infty} \frac{2}{x^3-x-17} = 0$, so this function has a horizontal asymptote at $y = 0$.
- $\lim_{x\to-\infty} e^{3x}+1 = 1$, so this function has a horizontal asymptote at $y = 1$.
- $\lim_{x\to\infty} \frac{x^3+x+71}{x^2-81} = \lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty$. Similarly, $\lim_{x\to-\infty} \frac{x^3+x+71}{x^2-81} = -\infty$. Therefore, this function has no horizontal asymptotes.

c. Which of the following functions have at least one *vertical* asymptote?

[✓ ○ $f(x) = \dfrac{1}{x-5}$]

[✓ ○ $f(x) = \ln(x)+5$]

○ $f(x) = \dfrac{x^2(x-1)^2}{(x-1)}$

○ $f(x) = \dfrac{x^2 - 2x + 5}{x^2 + 1}$

○ NONE OF THE ABOVE

Solution:

- $\frac{1}{x-5}$ has a vertical asymptote at $x = 5$.
- $\frac{x^2(x-1)^2}{(x-1)}$ is undefined at $x = 1$. However, since $x = 1$ occurs as a zero with multiplicity 1 in the denominator but multiplicity 2 in the numerator, it is a hole rather than a vertical asymptote. Therefore, this function has no vertical asymptotes.
- $\ln(x) + 5$ has a vertical asymptote at $x = 0$.
- Since $x^2 + 1$ has no (real) zeros, $\frac{x^2-2x+5}{x^2+1}$ is defined for all real numbers and has no vertical asymptotes.

d. In which of the following equations is y directly proportional to x^2?

○ $y = 2x$

○ $y = x^2 - 5$

[✓ ○ $y = 2x^2$]

[✓ ○ $y = \dfrac{\sqrt{7}x^2}{3}$]

○ $y = \dfrac{4}{x^2}$

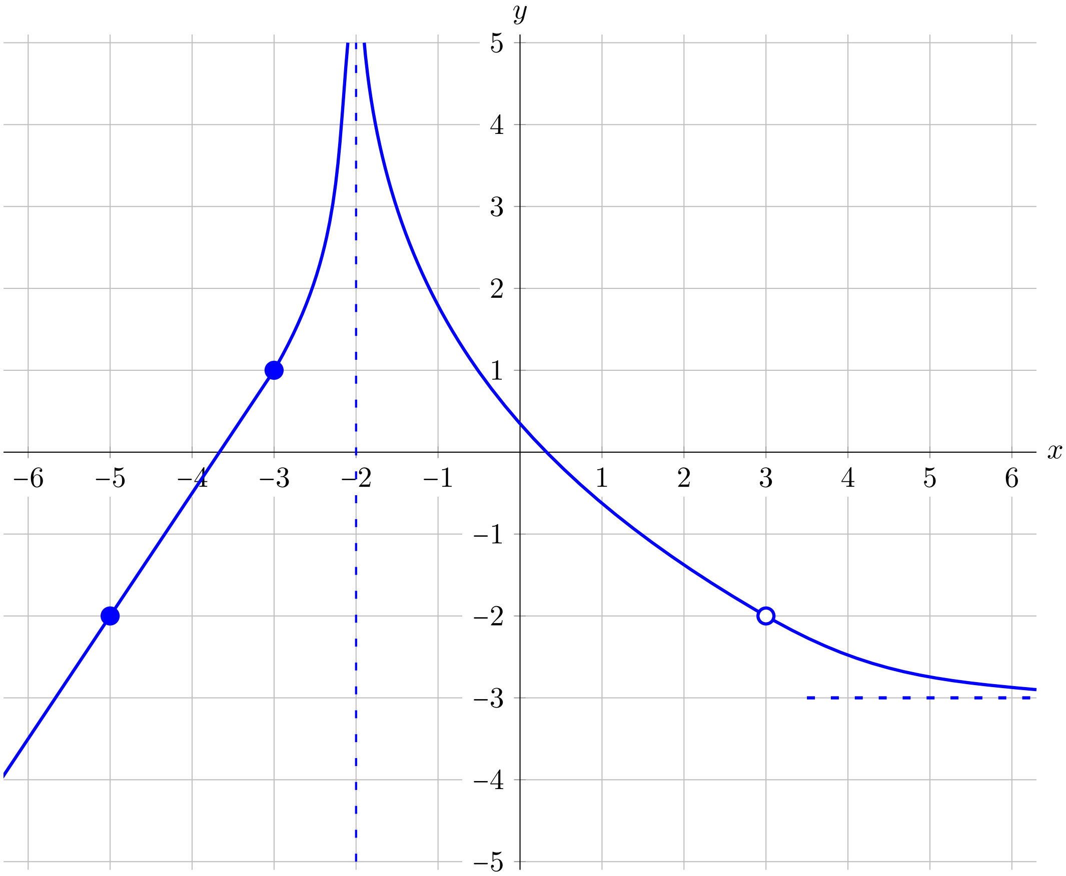
○ NONE OF THE ABOVE

Solution: We're looking for the options which have the form $y = kx^2$ for some real number k. The only options of this form are $y = 2x^2$ and $y = \frac{\sqrt{7}x^2}{3}$. If $y = \frac{4}{x^2}$, then y is inversely proportional to x^2, not directly proportional.

7. [7 points] On the axes below, sketch a graph of a **single function** $y = g(x)$ that satisfies all of the following properties:

- the domain of $g(x)$ is all real numbers except for -2 and 3
- the average rate of change of $g(x)$ on the interval $-5 \le x \le -3$ is $3/2$
- $g(x)$ has a vertical asymptote at $x = -2$
- $g(x)$ has a hole at $x = 3$
- $g(x)$ is concave up on the interval $(4, \infty)$
- $g(x)$ has a horizontal asymptote at $y = -3$

Solution: Below is one of many possible solutions.

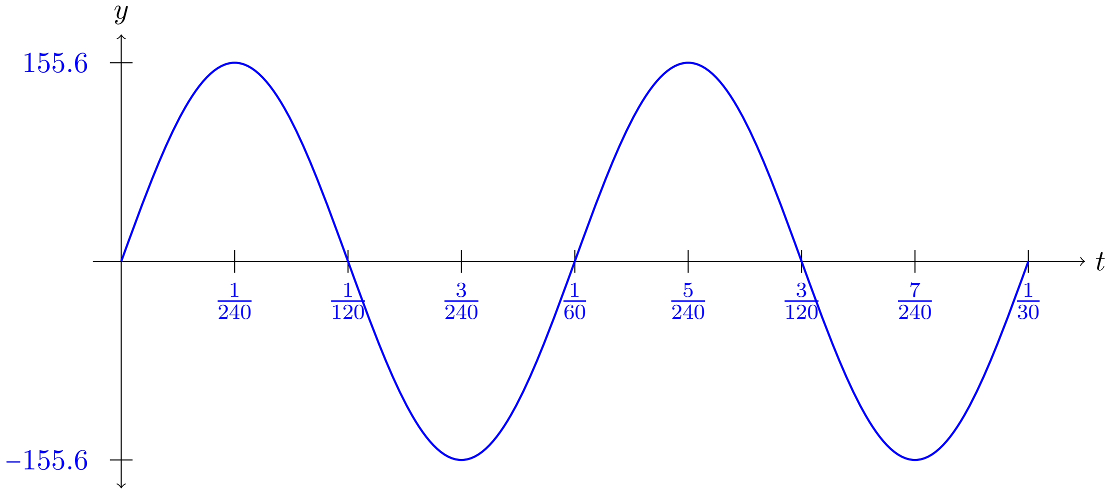


8. [11 points] In US households, electrical voltage (in volts) can be modeled by the function

$$V(t) = 155.6\sin(120\pi t)$$

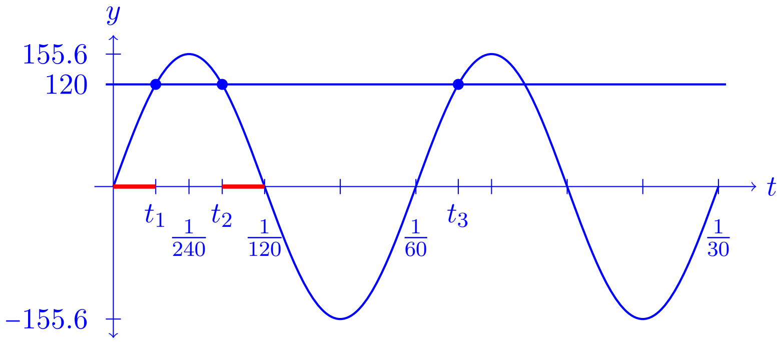
where t is measured in seconds.

a. [4 points] On the axes below, sketch a graph of **two periods** of $y = V(t)$. Your second cycle should end at exactly the furthest right tick on the t-axis. Clearly label at least two ticks on the t-axis. Use the ticks on the y-axis for your maximum and minimum values of $V(t)$ and label them as well.



b. [4 points] Find the first three positive values of t where the voltage is equal to 120 volts. *Show all work. Leave your answers in exact form or round to at least four decimal places.*

Solution: We want to find the first three positive values of t such that $V(t) = 155.6\sin(120\pi t) = 120$. We can mark the locations of these three solutions t_1, t_2, and t_3 on the graph we drew in part (a). We'll also label a few more tick marks to help us find the values of t_1, t_2, and t_3.



We first solve the equation $V(t) = 120$ for t algebraically:

$$\begin{aligned} 155.6\sin(120\pi t) &= 120 \\ \sin(120\pi t) &= \frac{120}{155.6} \\ 120\pi t &= \arcsin(120/155.6) \\ t &= \frac{\arcsin(120/155.6)}{120\pi} \approx 0.0023 \end{aligned}$$

We can check that this value of t is between 0 and $\frac{1}{240} \approx 0.0042$, so this must be the value of t_1. The third solution t_3 is then equal to $t_1 + 1/60 = \frac{\arcsin(120/155.6)}{120\pi} + \frac{1}{60} \approx 0.0190$ since the period of the function is $1/60$. To find the second solution, we note that the distance from 0 to t_1 is the same as the distance from t_2 to $\frac{1}{120}$. These two distances are shown in red in the graph above. This tells us that $t_1 - 0 = \frac{1}{120} - t_2$, so $t_2 = \frac{1}{120} - t_1 = \frac{1}{120} - \frac{\arcsin(120/155.6)}{120\pi} \approx 0.0060$

$t =$ $\frac{\arcsin(120/155.6)}{120\pi} \approx 0.0023$, $\frac{1}{120} - \frac{\arcsin(120/155.6)}{120\pi} \approx 0.0060$, $\frac{\arcsin(120/155.6)}{120\pi} + \frac{1}{60} \approx 0.0190$

c. [3 points] In Australia, the voltage alternates between a maximum of 240 volts, to a minimum of −240 volts, and back to 240 volts 50 times per second. Find a formula for the function $A(t)$ which models the voltage in Australia t seconds from when the voltage is at its **maximum**.

Solution: Since the voltage oscillates back and forth and starts at a maximum when $t = 0$, we will choose to express $A(t)$ as a transformation of $\cos(t)$. The amplitude must be 240, and the period must be $1/50$. Therefore $A(t) = 240\cos(100\pi t)$.

$A(t) =$ $240\cos(100\pi t)$

9. [7 points] Jay drinks a cup of coffee which contains 140mg of caffeine. The amount of caffeine, in milligrams, left in his body h hours after he drinks the coffee is given by the function

$$C(h) = 140(0.89)^h.$$

For each part below, show all work and leave your answer in exact form or rounded to at least two decimal places.

a. [2 points] What percent of the caffeine in the cup of coffee is left in Jay's body 2 hours after he drinks it?

Solution: After 2 hours, Jay has $C(2) = 140(0.89)^2$ mg of caffeine left in his body. This is $100 \cdot (0.89)^2\% = 79.21\%$ of the 140mg of caffeine that was in the cup of coffee.

79.21 %.

b. [2 points] What is the continuous percent decay rate of $C(h)$?

Solution: The growth factor of the exponential function $C(h)$ is 0.89. Therefore, the continuous decay rate as a decimal is $\ln(0.89) \approx -0.1165$. Therefore, the continuous percent decay rate is $100 \cdot (-\ln(0.89))\%$, or 11.65%.

$100 \cdot (-\ln(0.89)) \approx 11.65$ %

c. [3 points] Find the half-life of $C(h)$. *Include relevant units in your answer.*

Solution: We want to find the value of h such that $C(h) = 0.5C(0)$. Solving this equation for h gives us

$$\begin{aligned} C(h) &= 0.5C(0) \\ 140(0.89)^h &= 0.5(140)(0.89)^0 \\ 140(0.89)^h &= 0.5(140) \\ 0.89^h &= 0.5 \\ \log(0.89^h) &= \log(0.5) \\ h\log(0.89) &= \log(0.5) \\ h &= \frac{\log(0.5)}{\log(0.89)} \end{aligned}$$

The half-life of $C(h)$ is $\frac{\log(0.5)}{\log(0.89)} \approx 5.95$ **hours**.