Math 105 — First Midterm

February 7, 2012

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 9 pages including this cover. There are 8 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	12	
3	10	
4	11	
5	9	
6	14	
7	17	
8	13	
Total	100	

1. [14 points] Figure 1 below gives some data for an invertible function $f(x)$ and Figure 2 shows the graph of a function $g(x)$. Use this information to answer the questions below.

x	0	1	2	3	4	5	6
$f(x)$	1	5	8	9	7	4	0

Figure 1

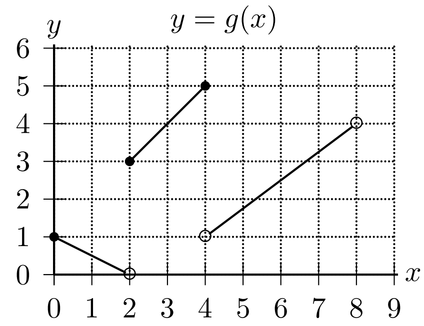

Figure 2

a. [4 points]

i. Evaluate $2g(2)$.

Solution: From the graph, we see that $g(2) = 3$. So $2g(2) = 2(3) = 6$.

Answer: 6

ii. Evaluate $f^{-1}(5)$.

Solution: From the table, we see that $f(1) = 5$. So $f^{-1}(5) = 1$.

Answer: 1

iii. Evaluate $f(f(1))$.

Solution: From the table, we see that $f(1) = 5$. So $f(f(1)) = f(5) = 4$.

Answer: 4

iv. Solve $f(x) = g(3)$ for x.

Solution: From the graph we see that $g(3) = 4$. So we are to solve $f(x) = 4$. From the table, we see that the solution is $x = 5$.

Answer: $x = 5$

b. [3 points] Which of the following numbers are in the *range* of g? (Circle ALL correct answers.)

0   [1]   [1.5]   [π]   [4]   [5]   5.25   7   8   9

c. [7 points] Find a formula for $g(x)$ as a piecewise-defined function.

Solution: The first piece appears to be linear with slope -0.5 and vertical intercept 1 so on this piece, $g(x) = 1 - 0.5x$. The second piece appears to be linear with slope 1 and vertical intercept 1, so on this piece, $g(x) = 1 + x$. The third piece appears to be linear with slope $3/4$, so using the point $(4, 1)$ and point-slope form, a formula for this piece is $g(x) = 1 + 0.75(x - 4) = -2 + 0.75x$. Hence a formula for $g(x)$ is

$$g(x) = \begin{cases} 1 - 0.5x & \text{if } 0 \le x < 2 \\ 1 + x & \text{if } 2 \le x \le 4 \\ -2 + 0.75x & \text{if } 4 < x < 8 \end{cases}$$

2. [12 points] A local grocery store sells dry goods in bulk, and one of the goods it sells is quinoa. It costs the store \$110.50 per month (for the space, employee time, etc.) to be able to stock and sell quinoa and \$1.25 per pound to purchase its supply of quinoa. The store charges customers \$4.50 per pound for quinoa.

a. [3 points] Let $C(q)$ be the monthly cost, in dollars, for the store to stock and sell q pounds of quinoa per month. Find a formula for $C(q)$.

Solution: Based on the given information, the average rate of change of the cost is constant (\$1.25 per pound), so $C(q)$ is linear with slope \$1.25/lb.
The fixed cost is \$110.50, so $C(q) = 110.50 + 1.25q$.

Answer: $C(q) =$ $110.50 + 1.25q$

b. [2 points] Let $R(q)$ be the store's monthly revenue from quinoa, in dollars, if it sells q pounds of quinoa that month. Find a formula for $R(q)$. *Recall that revenue is the total amount of money that the store brings in, i.e. how much money customers pay.*

Solution: The price for customers is \$4.50 per pound, so the revenue from selling q pounds is $R(q) = 4.50q$.

Answer: $R(q) =$ $4.50q$

c. [4 points] Assume that the store sells all of the quinoa that it buys each month. How many pounds of quinoa must the store sell in a month in order to not lose money from selling quinoa? (That is, how many pounds of quinoa must the store sell in order to break even on quinoa?) *Remember to show your work.*

Solution: The store will break even when $R(q) = C(q)$. Solving for q, we have

$$\begin{aligned} R(q) &= C(q) \\ 4.50q &= 110.50 + 1.25q \\ 3.25q &= 110.50 \\ q &= 110.50/3.25 \\ q &= 34 \end{aligned}$$

So the store breaks even when $q = 34$ (and makes a profit if $q > 34$). The store must sell at least 34 pounds of quinoa in order to not lose money from selling quinoa.

Answer: 34 **pounds**

d. [3 points] The store also sells almonds. Suppose it sells, on average, a_0 pounds of almonds per month. Let $P(a)$ be the profit, in dollars, that the store earns each month from selling a pounds of almonds. Give a practical interpretation of the quantity $P(a_0 + 100) - P(a_0)$. (*Include units. Your interpretation should not include any math symbols or variables.*)

Solution: The quantity $P(a_0 + 100) - P(a_0)$ is the additional monthly profit, in dollars, that the store earns from selling 100 more pounds of almonds in a month than they sell on average.

3. [10 points] The table below gives data about the popularity of some popular web browsers during 2011.[1]

- M is the month of the year. (So, for example, $M = 2$ represents February 2011.)
- F is the percent of internet users choosing Firefox.
- C is the percent of internet users choosing Chrome.
- S is the percent of internet users choosing Safari.

M	2	4	6	8	10
F	42.4	42.9	42.2	40.6	38.7
C	24.1	25.6	27.9	30.3	32.3
S	4.1	4.1	3.7	3.8	4.2

a. [5 points] Which, if any, of the statements below are supported by the data in the table above? (*Circle* ALL *such statements or circle* NONE OF THESE.)

S is a function of C. | F is a concave down function of M.

C is a function of S. | C is a concave up function of M.

F is a decreasing function of M. | C is a linear function of M.

C is an increasing function of M. | C is an exponential function of M.

NONE OF THESE

The popularity of another browser, Internet Explorer, is a function of the month M. Let $g(M)$ be the percent of all internet users who chose to use Internet Explorer in month M of 2011.

b. [2 points] Write an equation that expresses the fact that in January of 2011, 26.6% of internet users chose to use Internet Explorer as their internet browser.

Answer: $g(1) = 26.6$

c. [3 points] Let $B(p)$ be the amount, in dollars, of monthly bonuses paid to Internet Explorer programmers when p percent of internet users chose to use Internet Explorer. Interpret, in the context of this problem, the expression $B(g(2))$.
(*Use a complete sentence and include units.*)

Solution: $B(g(2))$ is the amount, in dollars, of the monthly bonuses paid to Internet Explorer programmers in February 2011.

[1]Source: http://www.w3schools.com/browsers/browsers_stats.asp

4. [11 points] Consider the line ℓ given by the equation $y = -3 + 0.2x$.

a. [3 points] Find the slope and both intercepts of ℓ.

Solution: Since the given equation is already in slope-intercept form, we see that the slope of ℓ is 0.2 and its y–intercept is -3. To find the x–intercept, we solve the equation $0 = -3 + 0.2x$ to find $x = 15$.

slope: 0.2 x**–intercept:** 15 y**–intercept:** -3

b. [3 points] Find an equation for the line that is perpendicular to the line ℓ (above) and passes through the point $(4, -2)$.

Solution: Since the slope of ℓ is 0.2, the slope of this line is $-1/0.2 = -5$. Using the given point $(4, -2)$ and point-slope form, we find $y + 2 = -5(x - 4)$ so $y = -2 - 5(x - 4)$. (Or, simplifying to slope-intercept form, this is $y = 18 - 5x$.)

Answer: $y =$ $-2 - 5(x - 4)$ **or** $y = 18 - 5x$

c. [5 points] Find an equation for the parabola satisfying *both* of the conditions below.

- Its y–intercept is 5.
- Its vertex is the point on the line ℓ (above) where $x = 10$.

Solution: The point on the line ℓ where $x = 10$ has y–coordinate $-3 + 0.2(10) = -1$. Hence $(10, -1)$ is the vertex of this parabola. Using vertex form, an equation is therefore given by $y = a(x - 10)^2 - 1$ for some non-zero constant a.

To find a we use the other piece of information provided.
The y–intercept of the parabola is 5, so the point $(0, 5)$ is on the parabola. Thus we have $5 = a(0 - 10)^2 - 1$ and, solving for a, we find $a = 0.06$. Hence an equation for the parabola is $y = 0.06(x - 10)^2 - 1$. (In standard form, this is $y = 0.06x^2 - 1.2x + 5$.)

Answer: $y =$ $0.06(x - 10)^2 - 1$

5. [9 points]

a. [3 points] The golden lion tamarin is an endangered species. However, due to conservation efforts, the number of wild golden lion tamarins has been increasing. There were 450 golden lion tamarins in the wild in 1990, and their population has grown by about 5.2% per year since then.[2] Let $L(y)$ be the number of wild golden lion tamarins y years after 1990. Find a formula for $L(y)$.

Solution: Since the population has been growing by a constant *percentage* per year, an appropriate model for $L(y)$ is exponential, say $L(y) = ab^y$. The information provided indicates that $a = 450$ and $r = 0.052$ so $b = 1.052$. Hence, a formula for $L(y)$ is $L(y) = 450(1.052)^y$.

Answer: $L(y) = 450(1.052)^y$

b. [3 points] The value of a typical new car decreases by about 9% per year. Find a formula for $V(t)$, the value of a car, in thousands of dollars, t years after purchase if its value when originally purchased was \$25,000.

Solution: Since the value decreases by a constant percentage each year, an appropriate model for $V(t)$ is exponential, so $V(t) = ab^t$. In this case, the information provided indicates that $a = 25$ (since $V(t)$ is measured in thousands of dollars) and $r = -0.09$ so $b = 0.91$. Hence a formula for $V(t)$ is $V(t) = 25(0.91)^t$.

Answer: $V(t) = 25(0.91)^t$

c. [3 points] Before concrete to pave a driveway starts to be poured, a concrete mixer contains 40,000 pounds of concrete. If paving 54 square inches of the driveway uses 25 pounds of concrete, find a formula for $C(d)$, the amount, in pounds, of concrete remaining in the mixer after d square inches of the driveway have been paved.

Solution: To pave the driveway, concrete is used at the constant rate of 25 pounds for every 54 square inches. Hence, $C(d)$ is a linear function with constant average rate of change $-25/54$ pounds per square inch. Since the amount of concrete in the mixer when 0 inches have been paved is 40000 pounds, we see that $C(d) = 40000 - \frac{25}{54}d$.

Answer: $C(d) = 40000 - \dfrac{25}{54}d$

[2]Source: http://www.animalinfo.org

6. [14 points] The number of internet users has increased dramatically since the internet was first introduced. There were 361 million internet users worldwide in December 2000 and 817 million internet users worldwide in December 2004. [3]

Let $U(t)$ be the number, in millions, of internet users worldwide t years after December 1997.

Remember to show your work carefully. All numbers appearing in your answers should either be in exact form or be accurate to at least three decimal places.

a. [3 points] Find the average rate of change of $U(t)$ between $t = 3$ and $t = 7$. Include units.

Solution: Note that $t = 7$ in December 2004 and $t = 3$ in December 2000. Hence $U(7) = 817$ and $U(3) = 361$.

The requested average rate of change is therefore $\dfrac{U(7) - U(3)}{7 - 3} = \dfrac{817 - 361}{4} = \dfrac{456}{4} = 114$ million internet users per year.

Answer: 114 **million internet users per year**

b. [4 points] Assuming that $U(t)$ is linear, find a formula for $U(t)$.

Solution: If $U(t)$ is linear, then its average rate of change is constant and from part (a) we know that this constant average rate of change of $U(t)$ is 114. Since $U(3) = 361$, we can use point-slope form to find $U(t) - 361 = 114(t - 3)$ or $U(t) = 361 + 114(t - 3)$. (Simplifying to slope-intercept form, this is $U(t) = 19 + 114t$, so $U(0) = 19$. Hence this model predicts that there were 19 million internet users in December 2007.)

Answer: $U(t) =$ $361 + 114(t - 3)$ **or** $19 + 114t$

According to this model, how many internet users were there in December 1997?

Answer: 19 **million**

c. [7 points] Assuming instead that $U(t)$ is exponential, find a formula for $U(t)$.

Solution: If $U(t)$ is exponential, then there are constants a and b so that $U(t) = ab^t$. Using the data provided in the problem statement, we know that $U(3) = 361$ and $U(7) = 817$. Hence $361 = ab^3$ and $817 = ab^7$. Dividing, we see that $\frac{817}{361} = \frac{ab^7}{ab^3}$ so $\frac{817}{361} = b^4$. Therefore $b = \left(\frac{817}{361}\right)^{1/4}$. Using this value of b in one of the equations above, we see that $361 = a\left[\left(\frac{817}{361}\right)^{1/4}\right]^3$ so $a = 361\left(\frac{361}{817}\right)^{3/4}$. Therefore, a formula for $U(t)$ is $U(t) = 361\left(\frac{361}{817}\right)^{3/4}\left[\left(\frac{817}{361}\right)^{1/4}\right]^t$. (This model thus predicts that there were $361\left(\frac{361}{817}\right)^{3/4}$ or about 195.646 million internet users in December 1997.)

Answer: $U(t) =$ $361\left(\frac{361}{817}\right)^{3/4}\left[\left(\frac{817}{361}\right)^{1/4}\right]^t \approx 195.6460(1.2265^t)$

According to this model, how many internet users were there in December 1997?

Answer: **Approximately** 195.646 **million**

[3]Source: http://www.internetworldstats.com

7. [17 points] Passengers on a cruise ship watch as nearby dolphins and porpoises jump through the waves. When one dolphin jumps, its height above the water (measured in feet) t seconds after leaving the water is given by $w(t) = -16t^2 + 96Dt$ for some positive constant D.

a. [3 points] After how many seconds, in terms of D, does this dolphin land back in the water?

Solution: The dolphin lands in the water when its height above the water is 0 ft. So we solve the equation $w(t) = 0$. Factoring, we see that $w(t) = -16t^2 + 96Dt = -16t(t - 6D)$ so $w(t) = 0$ when $t = 0$ and when $t = 6D$. Since $t = 0$ when the dolphin first leaves the water, $t = 6D$ when the dolphin lands back in the water. Hence the dolphin lands back in the water after $6D$ seconds.

Answer: $6D$ **seconds**

b. [7 points] Use the method of completing the square to rewrite $w(t)$ in vertex form. What is the vertex of the graph of $w(t)$? (Carefully show your work step-by step. Your answers may involve D.)

Solution:

$$\begin{aligned} w(t) &= -16t^2 + 96Dt \\ &= -16(t^2 - 6Dt) \\ &= -16(t^2 - 6Dt + (-3D)^2 - (-3D)^2) \\ &= -16\left[(t - 3D)^2 - 9D^2\right] \\ &= -16(t - 3D)^2 + 144D^2 \end{aligned}$$

Vertex Form: $w(t) =$ $-16(t - 3D)^2 + 144D^2$ **Vertex:** $(3D, 144D^2)$

c. [2 points] If the dolphin reaches a maximum height of 16 ft before falling back to the water, find the value of D.

Solution: The maximum height of the dolphin is the second coordinate of the vertex found above. So, $144D^2 = 16$ and thus $D^2 = 16/144 = 1/9$ so $D = \pm 1/3$. Since D is positive, $D = 1/3$.

Answer: $D =$ $1/3$

d. [5 points] A nearby porpoise is also seen jumping. Its height above the surface of the water (measured in meters) t seconds after the *dolphin* left the water is given by $h(t) = -5t^2 + 24t - 26$. For how long is the porpoise above the surface of the water? *Solve for the answer algebraically and give your final answer either in exact form or accurate to at least three decimal places.*

Solution: We solve the equation $h(t) = 0$ to find the time when the porpoise left and the time when the porpoise returned to the water. Using the quadratic formula, we have $-5t^2 + 24t - 26 = 0$ when $t = \frac{-24 \pm \sqrt{24^2 - 4(-5)(-26)}}{2(-5)} = \frac{-24 \pm \sqrt{56}}{-10} = \frac{-24 \pm 2\sqrt{14}}{-10} = \frac{12}{5} \pm \frac{\sqrt{14}}{-5}$. So, the porpoise leaves the water when $t = \frac{12}{5} - \frac{\sqrt{14}}{-5} \approx 1.6517$ and lands back in the water when $t = \frac{12}{5} - \frac{\sqrt{14}}{-5} \approx 3.1483$. Thus the porpoise is above the water for $(\frac{12}{5} - \frac{\sqrt{14}}{-5}) - (\frac{12}{5} + \frac{\sqrt{14}}{-5}) = \frac{2\sqrt{14}}{5} \approx 1.497$ seconds.

Answer: $\frac{2\sqrt{14}}{5} \approx 1.497$ **seconds**

8. [13 points] The *karat rating* of a gold alloy is defined to be 24 times the concentration of gold in the alloy. That is, the karat rating is $24 \cdot \dfrac{\text{mass of gold in alloy}}{\text{total mass of alloy}}$.

Rose gold is an alloy of gold and copper. A metallurgist is experimenting to see how the color of rose gold changes when more or less copper is added. In each trial, the metallurgist starts with 15 grams of gold and 5 grams of copper and then adds or removes copper to change the composition. Let $K(c)$ be the karat rating of the metallurgist's rose gold if c grams of copper have been added to ($c > 0$) or removed from ($c < 0$) the initial 5 grams.

a. [2 points] Find $K(0)$.

Solution: If no grams have copper have been added, then the rose gold consists of 15 grams of gold and 5 grams of copper, so $K(0) = 24 \cdot \frac{15}{20} = 18$. So the karat rating is 18.

Answer: $K(0) =$ 18

b. [5 points] In the context of this problem, what are the domain and range of $K(c)$? (You may use either interval notation or inequalities to describe the domain and range.) *Show your work and explain your reasoning.*

Solution: Any amount of copper can be added to the sample, but at most 5 grams can be removed. Therefore, in the context of this problem, c can be any number in the interval $[-5, \infty)$. Hence the domain of $K(c)$ is $[-5, \infty)$.
The concentration of gold in the alloy can be anywhere from 0% to 100%, not including 0% since gold is never removed from the sample, but including 100% since the metallurgist could remove all 5 grams of copper. Hence the karat rating of the sample can be anywhere from 0 to 24 (not including 0, but including 24). In other words, the range of $K(c)$ is $(0, 24]$.

Domain: $[-5, \infty)$ **Range:** $(0, 24]$

c. [3 points] Find a formula for $K(c)$.

Solution: The total mass of gold in the sample is fixed at 15 grams, but the total mass of the alloy after c grams of copper have been added or removed is $20 + c$. Thus, a formula for $K(c)$ is $K(c) = 24\dfrac{15}{20+c} = \dfrac{360}{20+c}$

Answer: $K(c) = \dfrac{360}{20+c}$

One variety of *white gold* is an alloy of gold and nickel.
Let $V(k)$ be the value, in dollars per gram, of k karat white gold.

d. [3 points] Give an interpretation, in the context of this problem, of the equation $V^{-1}(14) = 10$. *Use a complete sentence and include units.*

Solution: White gold that has a value of 14 dollars per gram is 10 karat.