Math 105 — Second Midterm

March 20, 2012

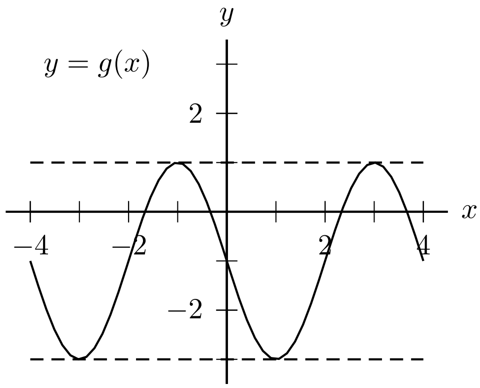
Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	14	
2	12	
3	5	
4	7	
5	12	
6	13	
7	10	
8	8	
9	9	
10	10	
Total	100	

1. [14 points] *Note: No work or explanation is required on this page.*
The graph of a sinusoidal function g is shown below.



a. [6 points] Find the period, amplitude, and midline of $y = g(x)$.

Period: 4

Amplitude: 2

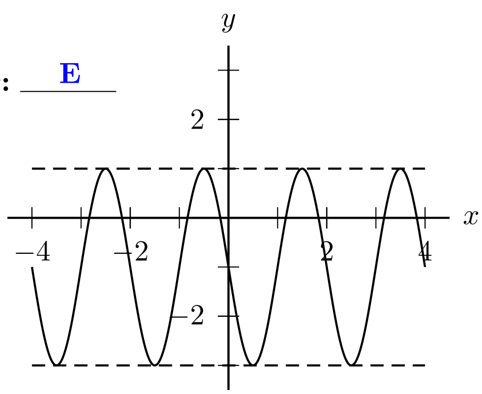
Midline: $y = -1$

b. [8 points] Below are the graphs of several transformations of $g(x)$. For each of these graphs, write the letter of the ONE function from the "Answer Choices" whose graph is shown. (**Clearly** write the capital letter of your choice on the answer blank provided.)

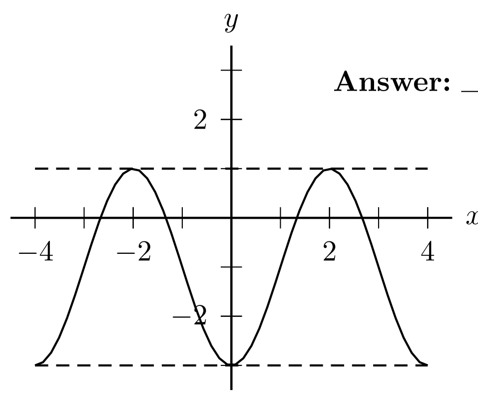
Answer Choices

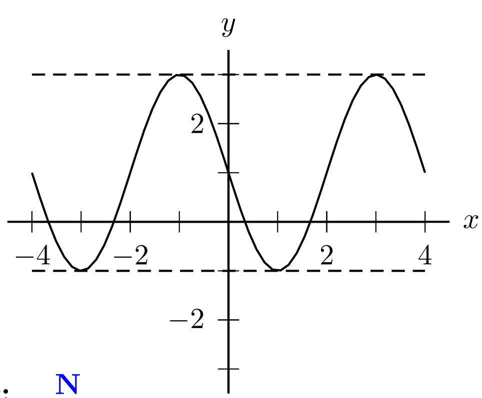
A.	$g(\pi x)$	E.	$g(2x)$	I.	$g(x-1)$	M.	$g(x-2)$
B.	$\pi g(x)$	F.	$g(\frac{1}{2}x)$	J.	$g(x)+1$	N.	$g(x)+2$
C.	$\frac{1}{2}g(x)$	G.	$g(x)-1$	K.	$g(x)-2$	O.	$2g(x)-\frac{1}{2}$
D.	$2g(x)$	H.	$g(x+1)$	L.	$g(x+2)$	P.	$\frac{1}{2}g(x)+\frac{1}{2}$

Answer: E

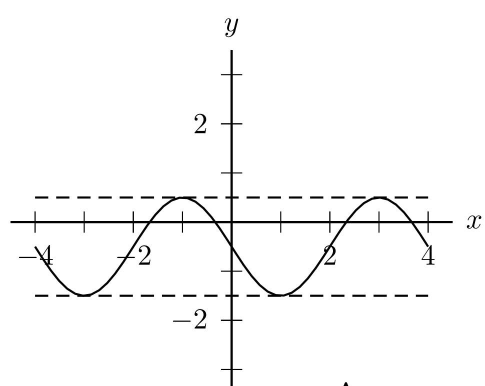


Answer: H





Answer: N



Answer: C

2. [12 points] Solve each of the equations below.
Show your work step-by-step and give the **exact solutions** *in the answer blanks provided.*

a. [3 points] $4(3.2)^t = 7$

Solution: Dividing both sides of the equation by 4 we find $(3.2)^t = 1.75$. We then use logarithms to find t.

$$\begin{aligned} \log((3.2)^t) &= \log(1.75) \\ t\log(3.2) &= \log(1.75) \\ t &= \frac{\log(1.75)}{\log(3.2)} \end{aligned}$$

Answer: $t =$ $\frac{\log(1.75)}{\log(3.2)}$

b. [3 points] $3e^{\ln(x+2)} = 8$

Solution: Dividing both sides of the equation by 3, we have $e^{\ln(x+2)} = \frac{8}{3}$. Since $e^{\ln(x+2)} = x + 2$, we find that $x + 2 = \frac{8}{3}$ so $x = \frac{8}{3} - 2 = \frac{2}{3}$.

Answer: $x =$ $\frac{2}{3}$

c. [3 points] $e^{m+5} = 6e^{-3m}$

Solution:
We first take the natural logarithm of both sides of the equation. Using properties of logarithms we can then solve for m.

$$\begin{aligned} \ln(e^{m+5}) &= \ln(6e^{-3m}) \\ \ln(e^{m+5}) &= \ln(6) + \ln(e^{-3m}) \\ m + 5 &= \ln(6) - 3m \\ 4m &= \ln(6) - 5 \\ m &= \frac{\ln(6) - 5}{4} \end{aligned}$$

Answer: $m =$ $\frac{\ln(6) - 5}{4}$

d. [3 points] $\ln(y + 3) - \ln(1 - y) = \ln(6)$

Solution: We rewrite the left side of the equation as $\ln\left(\frac{y+3}{1-y}\right)$. Then $\ln\left(\frac{y+3}{1-y}\right) = \ln(6)$ so exponentiating we have $e^{\ln\left(\frac{y+3}{1-y}\right)} = e^{\ln(6)}$ and hence $\frac{y+3}{1-y} = 6$. Then $y + 3 = 6(1 - y)$ so $y + 3 = 6 - 6y$. Hence $7y = 3$ and finally $y = 3/7$.

Answer: $y =$ $\frac{3}{7}$

3. [5 points] A mysterious substance decays by 30% every 6 years. Find the half-life of this substance. (*Show your work carefully and either give your answer in exact form or round your answer to the nearest* 0.01 *year.*)

Solution: Let $Q(t)$ be the quantity of the mysterious substance in year t and let a be the initial quantity. Then $Q(t) = ae^{kt}$ for some constant k.
Since the substance decays by 30% every 6 years, $Q(6) = 0.7a$ so $0.7a = ae^{6k}$. Then

$$\begin{aligned} 0.7a &= ae^{6k} \\ 0.7 &= e^{6k} \\ \ln(0.7) &= 6k \\ k &= \ln(0.7)/6. \end{aligned}$$

If h is the half-life of the substance, then $Q(h) = 0.5a$, so we have $0.5a = ae^{kh}$. Using the value of k we found above, this gives $0.5a = ae^{h\ln(0.7)/6}$ and we can solve for h.

$$\begin{aligned} 0.5a &= ae^{h\ln(0.7)/6} \\ 0.5 &= e^{h\ln(0.7)/6} \\ \ln(0.5) &= \ln\left(e^{h\ln(0.7)/6}\right) \\ \ln(0.5) &= h\ln(0.7)/6 \\ 6\ln(0.5)/\ln(0.7) &= h. \end{aligned}$$

So the half-life of this mysterious substance is $\dfrac{6\ln(0.5)}{\ln(0.7)}$ (or about 11.66) years.

Answer: $\dfrac{6\ln(0.5)}{\ln(0.7)} \approx 11.66$ **years**

4. [7 points] Consider the function B defined by $B(x) = 15 - e^{-0.001x}$.

a. [3 points] Let $f(x) = e^x$. Use transformations to find a formula for $B(x)$ in terms of f.

$B(x) =$ $-f(-0.001x) + 15$

b. [4 points] Find the vertical and horizontal asymptotes of the graph of $y = B(x)$. (If there are no vertical or no horizontal asymptotes, write "NONE" on the appropriate line(s).)

Solution: The function $f(x) = e^x$ from part (a) has no vertical asymptotes and has the horizontal asymptote $y = 0$. The graph of $y = B(x)$ is obtained from that of $y = f(x)$ by first stretching horizontally away from the y–axis by a factor of 1000 then reflecting over both the x– and y– axes and finally shifting up by 15 units. The resulting graph still has no vertical asymptote and has a horizontal asymptote of $y = 15$.

Vertical asymptote(s): **None**

Horizontal asymptote(s): $y = 15$

5. [12 points] *Note: You do not have to show any work on this page.*

a. [6 points] If $(2, -6)$ is a point on the graph of $y = h(x)$, find a point on the graph of each of the functions below.

(i) $\left(\underline{\quad 1 \quad}, \underline{\quad -6 \quad}\right)$ is a point on the graph of $y = h(2x)$.

Solution: To obtain the graph of $y = h(2x)$ from the graph of $y = h(x)$ we compress horizontally towards the y–axis by a factor of $1/2$, moving $(2, -6)$ to the point $(-1, 6)$.

(ii) $\left(\underline{\quad -2 \quad}, \underline{\quad -5 \quad}\right)$ is a point on the graph of $y = h(-x) + 1$.

Solution: To obtain the graph of $y = h(-x) + 1$ from the graph of $y = h(x)$, we first reflect the graph across the y–axis (moving the point $(2, -6)$ to the point $(-2, -6)$) and then shift the resulting graph up by one unit (moving the point $(-2, -6)$ to the point $(-2, -5)$).

(iii) $\left(\underline{\quad 3 \quad}, \underline{\quad 18 \quad}\right)$ is a point on the graph of $y = -3h(x - 1)$.

Solution: To obtain the graph of $y = -3h(x - 1)$ from the graph of $y = h(x)$, we first stretch the graph vertically away from the x–axis by a factor of 3 (moving the point $(2, -6)$ to the point $(2, -18)$). Then we reflect this graph across the x–axis (moving the point $(2, -18)$ to the point $(2, 18)$). Finally, we shift the resulting graph to the right by one unit (moving the point $(2, 18)$ to the point $(3, 18)$).

b. [6 points] Some data for functions g and k is provided in the table below. Use this data to answer the questions that follow.

x	1	2	3
$g(x)$	4	-1	-2
$k(x)$	5	4	1

(i) If $g(x)$ is an even function, find $g(-2)$.

Solution: Since g is even, $g(-2) = g(2)$ and the table indicates that $g(2) = -1$. Hence $g(-2) = -1$.

Answer: $g(-2) = \underline{\quad -1 \quad}$

(ii) Let $m(t) = 2k(-t + 1)$. Find $m(-2)$.

Solution: Using the provided formula for $m(t)$ and the given table, we have $m(-2) = 2k(-(-2) + 1) = 2k(2 + 1) = 2k(3) = 2(1) = 2$.

Answer: $m(-2) = \underline{\quad 2 \quad}$

(iii) Let $n(x) = k(x - 1)$. If $n(x)$ is an odd function, find $k(-3)$.

Solution: Using the given formula (or the implication that the graph of $y = n(x)$ results from shifting the graph of $y = k(x)$ to the right one unit), we have $k(-3) = n(-2)$. Now n is odd so $n(-2) = -n(2)$. Using the formula provided, we thus find that $k(-3) = n(-2) = -n(2) = -k(2 - 1) = -k(1) = -5$. Hence $k(-3) = -5$.

Answer: $k(-3) = \underline{\quad -5 \quad}$

6. [13 points] A study of mammals in a particular county in Michigan found that at the time of the study there were N groundhogs and that the population of groundhogs was increasing at a rate of 5% per year. Let $G(t)$ be the number of groundhogs in the county t years after the study.

For full credit on this problem, you must solve for all answers algebraically and show all work step-by-step. Answers should either be in exact form or be given to at least four decimal places.

a. [2 points] Find a formula for $G(t)$.

Answer: $G(t) =$ $N(1.05)^t$.

b. [3 points] Find the *continuous* growth rate of the groundhog population.

Solution: In the two standard forms for exponential functions ($Q = ab^t$ and $Q = ae^{kt}$) we have $b = e^k$ so $\ln b = k$. Here, b is the growth factor and k is the continuous growth rate. In the case of these groundhogs, the annual growth factor is 1.05, so the continuous growth rate is $\ln(1.05)$ (or about 4.8790%) per year.

Answer: $\ln(1.05)$ **(or about** 4.8790%**) per year**

c. [3 points] How long will it take for the number of groundhogs to double?

Solution: If d is the number of years it takes for the number of groundhogs to double, then $G(d) = 2N$. Hence we have $2N = N(1.05)^d$ so $2 = (1.05)^d$. Taking the natural logarithm of both sides of this equation, we find that $\ln(2) = \ln((1.05)^d) = d\ln(1.05)$. Hence $d = \ln(2)/\ln(1.05)$. So, it takes $\ln(2)/\ln(1.05)$ (about 14.2067) years for the number of groundhogs to double.

Answer: $\ln(2)/\ln(1.05)$ **(or about** 14.2067**) years**

d. [5 points] In the same study, it was determined that the number of moles and rabbits in the county t years after the study would be given by the formulas $M(t) = 500(0.99)^t$ and $R(t) = 200e^{0.1t}$, respectively. According to these models, when will the population of rabbits be 50% larger than the population of moles?

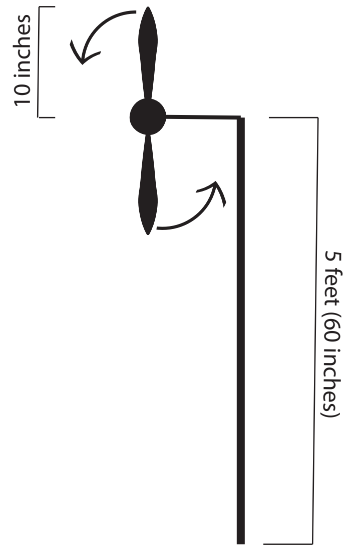
Solution: The population of rabbits will be 50% larger than the population of moles when $R(t) = 1.5M(t)$. So we need to find t so that $200e^{0.1t} = 1.5(500(0.99)^t)$. Using the definition and basic properties of the natural logarithm, we find

$$\begin{aligned}
200e^{0.1t} &= 1.5(500(0.99)^t) \\
e^{0.1t} &= 3.75(0.99)^t \\
0.1t &= \ln(3.75(0.99)^t) = \ln(3.75) + \ln((0.99)^t) = \ln(3.75) + t\ln(0.99) \\
0.1t - t\ln(0.99) &= \ln(3.75) \\
t(0.1 - \ln(0.99)) &= \ln(3.75) \\
t &= \ln(3.75)/(0.1 - \ln(0.99))
\end{aligned}$$

Therefore, the population of rabbits will be 50% greater than the population of moles $\ln(3.75)/(0.1 - \ln(0.99))$ (or about 12.0105) years after the study.

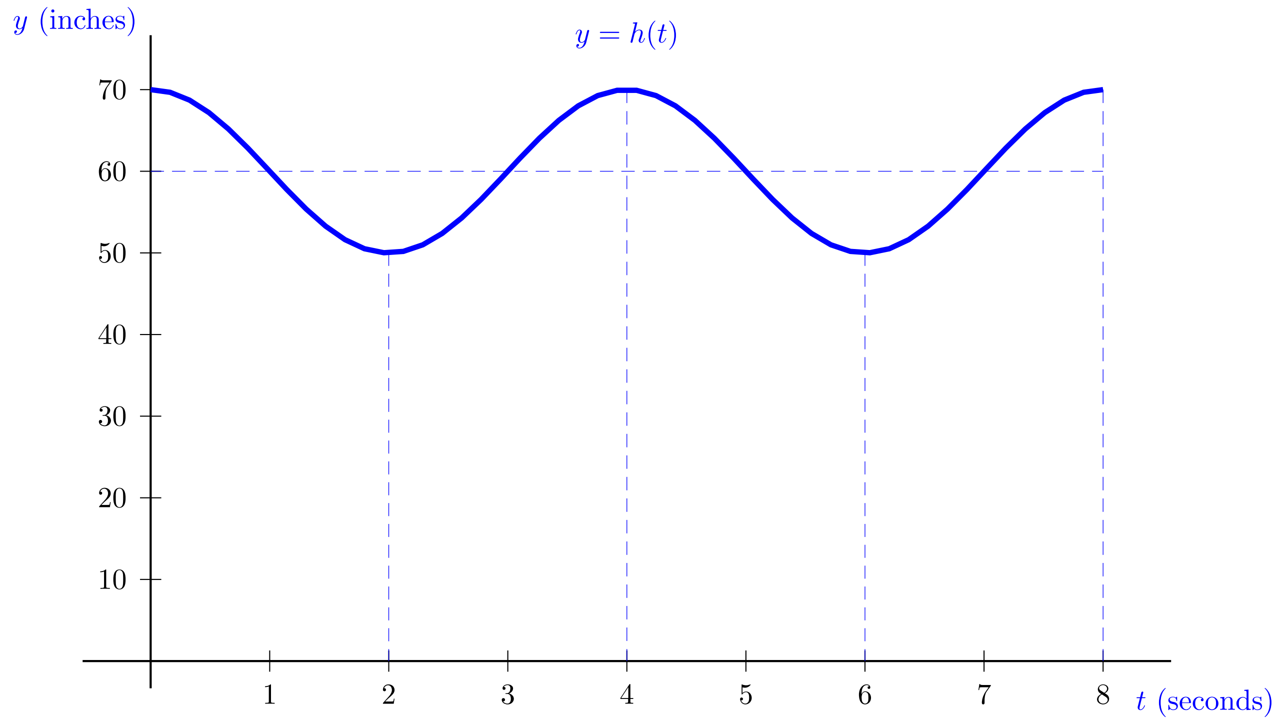
Answer: $\ln(3.75)/(0.1 - \ln(0.99))$ **(or about** 12.0105**) years after the study**

7. [10 points] A "Whirlydoodle"[1] is a small windmill that spins and lights up when the wind blows. One evening, there is a light breeze and a particular Whirlydoodle's blades are rotating at a constant rate of one revolution every 4 seconds. A moth lands on the tip of one of the blades of the Whirlydoodle when the blade is pointed straight up. (The moth then hangs on and rides for a minute.) This Whirlydoodle is mounted 5 feet (60 inches) above the ground, and each blade is 10 inches long, as shown in the diagram on the right.



Let $h(t)$ be the height (in inches) of the moth above the ground t seconds after the moth lands on the Whirlydoodle.

a. [6 points] Sketch a graph of $y = h(t)$ for $0 \le t \le 8$. (Remember to label the axes (including units) and to make sure that the key features and characteristics of your graph are clear.)



b. [4 points] Find a formula for $h(t)$.

Solution: h is a sinusoidal function with period 4 seconds. The graph of $y = h(t)$ has amplitude 10 (inches) and midline $y = 60$. (Since the function attains a maximum at $t = 0$, it is convenient to write $h(t)$ using transformations of the function $\cos t$. However, there are many other possible answers.)

Answer: $h(t) = $ $10\cos\left(\frac{\pi}{2}t\right) + 60$

[1] "Whirlydoodles" can be seen around downtown Ann Arbor.

8. [8 points] In the 1970's, seismologists developed the Moment Magnitude Scale (MMS) to estimate the magnitude of large earthquakes in terms of the energy released. Unlike the Richter scale, which is based on the size of seismic waves, the MMS is based on seismic moments (which represent the energy released in an earthquake). The MMS rating of an earthquake is defined to be

$$S = \frac{2}{3}\log\left(\frac{M}{A}\right)$$

where M is the seismic moment of the quake (in dynes/cm) and A is a positive constant.

a. [4 points] Let S_1 and S_2 represent the MMS ratings of two earthquakes with seismic moments M_1 and M_2, respectively. Using properties of logarithms, find a formula for $S_2 - S_1$ in terms of M_1 and M_2. **Simplify your formula as much as possible.**

Solution: Using the formula provided and basic properties of logarithms, we have

$$\begin{aligned} S_2 - S_1 &= \frac{2}{3}\log\left(\frac{M_2}{A}\right) - \frac{2}{3}\log\left(\frac{M_1}{A}\right) \\ &= \frac{2}{3}\left(\log\left(\frac{M_2}{A}\right) - \log\left(\frac{M_1}{A}\right)\right) \\ &= \frac{2}{3}\log\left(\frac{M_2/A}{M_1/A}\right) \\ &= \frac{2}{3}\log\left(\frac{M_2}{M_1}\right) \end{aligned}$$

Answer: $S_2 - S_1 =$ $\frac{2}{3}\log\left(\frac{M_2}{M_1}\right)$.

b. [4 points] The San Francisco earthquake of 1989 had an MMS rating of 6.9 and the Northridge, CA earthquake of 1994 had an MMS rating of 6.7. Based on these ratings, how many times greater than the Northridge seismic moment was the San Francisco seismic moment? (Give your answer in exact form or round to the nearest 0.01.)

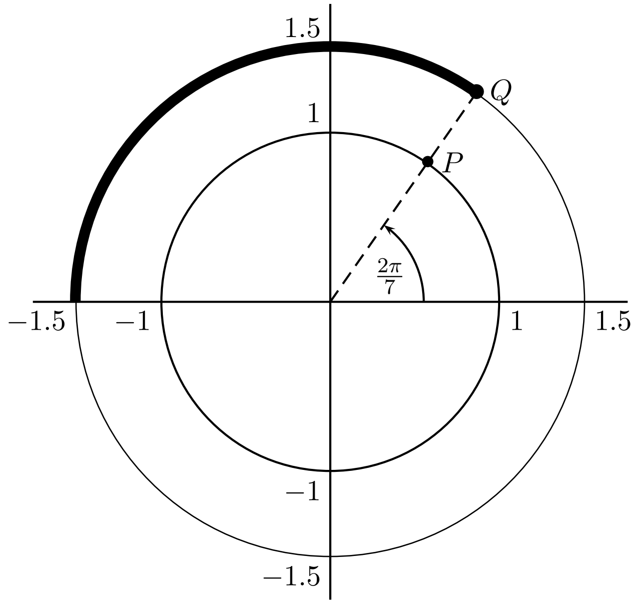
Solution: Let M_1 and M_2 be the seismic moments of the Northridge and San Francisco earthquakes, respectively. We are asked to find M_2/M_1.
Using the formula we found in part (a) (with $S_2 = 6.9$ and $S_1 = 6.7$), we have

$$\begin{aligned} 6.9 - 6.7 &= \frac{2}{3}\log\left(\frac{M_2}{M_1}\right) \\ 0.2 &= \frac{2}{3}\log\left(\frac{M_2}{M_1}\right) \\ 0.3 &= \log\left(\frac{M_2}{M_1}\right) \\ 10^{0.3} &= \frac{M_2}{M_1} \end{aligned}$$

Therefore, $M_2 = 10^{0.3}M_1$, so the seismic moment of the San Francisco earthquake was $10^{0.3}$ (about 2.00) times greater than that of the Northridge earthquake.

Answer: The seismic moment from the San Francisco earthquake was $10^{0.3}$ **(or about** 2.00**)** times greater than the seismic moment of the Northridge earthquake.

9. [9 points] Consider the points P and Q determined by the angle $\frac{2\pi}{7}$ as shown in the diagram below.



You don't have to show work, but any work you do show may be considered for partial credit. **Give all answers in *exact form*.**

a. [2 points] Find the coordinates of the point P.

Answer: The coordinates of P are ($\cos(2\pi/7)$, $\sin(2\pi/7)$).

b. [2 points] Find the coordinates of the point Q.

Answer: The coordinates of Q are ($1.5\cos(2\pi/7)$, $1.5\sin(2\pi/7)$).

c. [2 points] Find the length of the counterclockwise path from the point Q to the point $(-1.5, 0)$. (This path is shown in **bold** in the diagram above.)

Solution: The angle spanned by the path is $\pi - \frac{2\pi}{7} = \frac{5\pi}{7}$ radians. Hence the arclength is $1.5\left(\frac{5\pi}{7}\right) = \frac{7.5\pi}{7}$ units.

Answer: $\frac{7.5\pi}{7}$ **units**

d. [3 points] An ant begins at the point P, walks *clockwise* along the unit circle for 3 units and then stops. What are the coordinates of the point at which the ant stops?

Solution: Walking along the *unit* circle for 3 units corresponds to walking along an arc spanned by an angle measuring 3 radians. Since the ant walks *clockwise*, the final coordinates of the ant are thus the coordinates of the point on the unit circle determined by the angle $\frac{2\pi}{7} - 3$.

Answer: The coordinates of this point are ($\cos\left(\frac{2\pi}{7} - 3\right)$, $\sin\left(\frac{2\pi}{7} - 3\right)$).

10. [10 points] Ivanka is a student at a nearby college. Let $C(h)$ be the total tuition, in thousands of dollars, the college charges her if she takes h credit hours, and let a be the average number of credit hours students take at the college.
For each of the following, pick the ONE expression from the list of "Answer Choices" that best represents the described quantity. Clearly write the capital letter of your choice on the answer blank provided.

Answer Choices

A. $C(3)$	F. $C(a)+3$	K. $3C^{-1}(a)$	P. $C^{-1}(3a)$
B. $C^{-1}(3)$	G. $C(a-3)$	L. $C^{-1}(a)/3$	Q. $3C(C(a))$
C. $C(a)$	H. $C(a)-3$	M. $C(a)/3$	R. $C(3C^{-1}(a))$
D. $C^{-1}(a)$	I. $3C(a)$	N. $C(a/3)$	S. $C^{-1}(3C(a))$
E. $C(a+3)$	J. $C(3a)$	O. $C^{-1}(a/3)$	T. $C^{-1}(C(3a))$

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a. [2 points] Ivanka's tuition (in thousands of dollars) if she takes a total of 3 credit hours

Answer: **A.** $C(3)$

b. [2 points] Ivanka's total tuition (in thousands of dollars) if she takes 3 credit hours more than average

Answer: **E.** $C(a+3)$

c. [2 points] Ivanka's tuition (in thousands of dollars) if she takes one third the average number of credit hours

Answer: **N.** $C(a/3)$

d. [2 points] The amount (in thousands of dollars) that Ivanka pays for tuition if she takes the average number of credit hours but has a scholarship that covers three thousand dollars of her tuition

Answer: **H.** $C(a)-3$

e. [2 points] The number of credit hours Ivanka takes if her total tuition is three times as much as the tuition for taking the average number of credit hours

Answer: **S.** $C^{-1}(3C(a))$