Math 105 — Final Exam

April 19, 2012

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	11	
3	15	
4	7	
5	6	
6	5	
7	11	
8	5	
9	4	
10	11	
11	9	
12	8	
Total	100	

1. [8 points] For each of the statements below, circle "**True**" if the statement is *always* true. Otherwise circle "**False**". You do not need to show any work for this problem.

a. [2 points] All even degree polynomials are even functions.

True $\boxed{\text{False}}$

b. [2 points] If k is a positive constant, then the graph of $y = k \ln x$ is concave down.

$\boxed{\text{True}}$ False

c. [2 points] If $g(x) = e^x$ then $g^{-1}(x) = e^{-x}$ for all values of x.

True $\boxed{\text{False}}$

d. [2 points] As $x \to \infty$, the function $m(x) = (1.05)^x$ dominates the function $d(x) = 419x^{2012}$.

$\boxed{\text{True}}$ False

2. [11 points] *For full credit on this problem, you must show your work carefully. Unless specified otherwise, answers should either be in exact form or be rounded accurately to at least three decimal places.*

The population of Linearville grew from 9,000 in January 2003 to 14,000 in January 2005.

a. [2 points] Find the average rate of change of the population of Linearville between January 2003 and January 2005. (*Include units.*)

Solution: Since the population grew by 5,000 people in the two years from January 2003 to January 2005, the average rate of change is $\frac{5000}{2} = 2500$ people per year.

Answer: 2500 **people per year**

b. [3 points] The population of Linearville has been growing linearly since January 2000. Find a formula for $L(t)$, the population of Linearville t years after January 2000.

Solution: The average rate of change from part (a) gives the slope of $L(t)$. Since the population was 9,000 in January 2003, we see that $L(3) = 9000$. Using point-slope form, we find that $L(t) = 9000 + 2500(t - 3)$ or $L(t) = 1500 + 2500t$.

Answer: $L(t) =$ $1500 + 2500t$

The neighboring town of Exponential Corner has also been growing. Its population was 100 in January 2003 and had risen to 150 by January 2005.

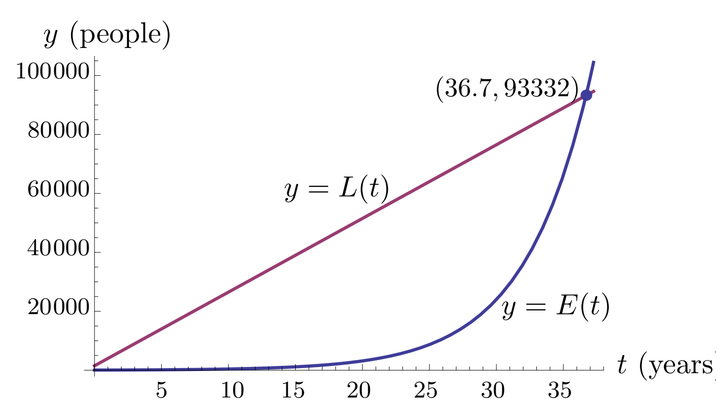
c. [4 points] Suppose that since January 2000, the population of Exponential Corner has been growing exponentially. Find a formula for $E(t)$, the population of Exponential Corner t years after January 2000.

Solution: Since the function is exponential, a formula for $E(t)$ is given by $E(t) = ab^t$ for some constants a and b. We are told that $E(3) = 100$ and $E(5) = 150$, so we find that $100 = ab^3$ and $150 = ab^5$. Dividing, we see that $\frac{150}{100} = \frac{ab^5}{ab^3}$ so $1.5 = b^2$. Then $b = \sqrt{1.5} \approx 1.225$ and we can solve for a. In particular $100 = a(\sqrt{1.5})^3$ so $a = \frac{100}{\sqrt{1.5}^3} \approx 54.433$. Thus $E(t) = \frac{100}{(\sqrt{1.5})^3}(\sqrt{1.5})^t \approx 54.433(1.225)^t$.

Answer: $E(t) = \frac{100}{(\sqrt{1.5})^3}(\sqrt{1.5})^t \approx 54.433(1.225)^t$

d. [2 points] Assuming the populations continue to grow as described above, will the population of Exponential Corner ever catch up to the population of Linearville?
If so, when will this happen? (*Round to the nearest year.*)
If not, explain how you know this.

Solution: Since $E(t)$ is an increasing positive exponential function, it will eventually dominate the linear function $L(t)$. Using a graphing calculator's "intersect" feature, we find that $L(t) = E(t)$ when $t \approx 36.7$. So, the population of Exponential Corner will catch up to that of Linearville in late 2036 at which time both populations will be about 93,332.



3. [15 points] *Show your work.*

a. [3 points] Find an equation for the straight line passing through the point $(2, -3)$ that is perpendicular to the line passing through the points $(1, 4)$ and $(-6, 5)$.

Solution: The slope of the line passing through $(1, 4)$ and $(-6, 5)$ is $\frac{5-4}{-6-1} = -\frac{1}{7}$, so the slope of a line perpendicular to it is 7. Using point-slope form, we therefore find that $y + 3 = 7(x - 2)$ describes the perpendicular line passing through the point $(2, -3)$.

Answer: $y =$ $7(x-2) - 3$ **or** $-17 + 7x$

b. [3 points] A population of ants is growing by 25% per day. How long will it take for the number of ants to double? (*Give your answer in exact form or rounded accurately to three decimal places.*)

Solution: The population can be modeled by an exponential function of the form $a(1.25)^t$ where t is measured in days and a is the population on day $t = 0$. If d is the doubling time, then $2a = a(1.25)^d$ so $2 = 1.25^d$. Solving for d we have $\ln 2 = d \ln 1.25$ so $d = \dfrac{\ln 2}{\ln 1.25} \approx 3.106$.

Answer: $d = \dfrac{\ln 2}{\ln 1.25} \approx 3.106$ **days**

c. [3 points] An ant begins at the point $(1, 0)$ and walks *counterclockwise* along the unit circle for a distance of 2 units and then stops. What are the coordinates of the point at which the ant stops? (*Give your answer in exact form.*)

Solution: On the unit circle, a distance of 2 units corresponds to an angle of 2 radians, so this counterclockwise walk starts at $(1, 0)$ and ends at the point $(\cos 2, \sin 2)$.

Answer: $(\cos 2, \sin 2)$

d. [3 points] Suppose the graph of $y = h(x)$ is obtained from the graph of $y = 3e^{2x}$ by shifting the graph of $y = 3e^{2x}$ to the right four units and then down five units.
Find a formula for $h(x)$.

Solution: Let $f(x) = 3e^{2x}$. Then $h(x) = f(x - 4) - 5 = 3e^{2(x-4)} - 5$.

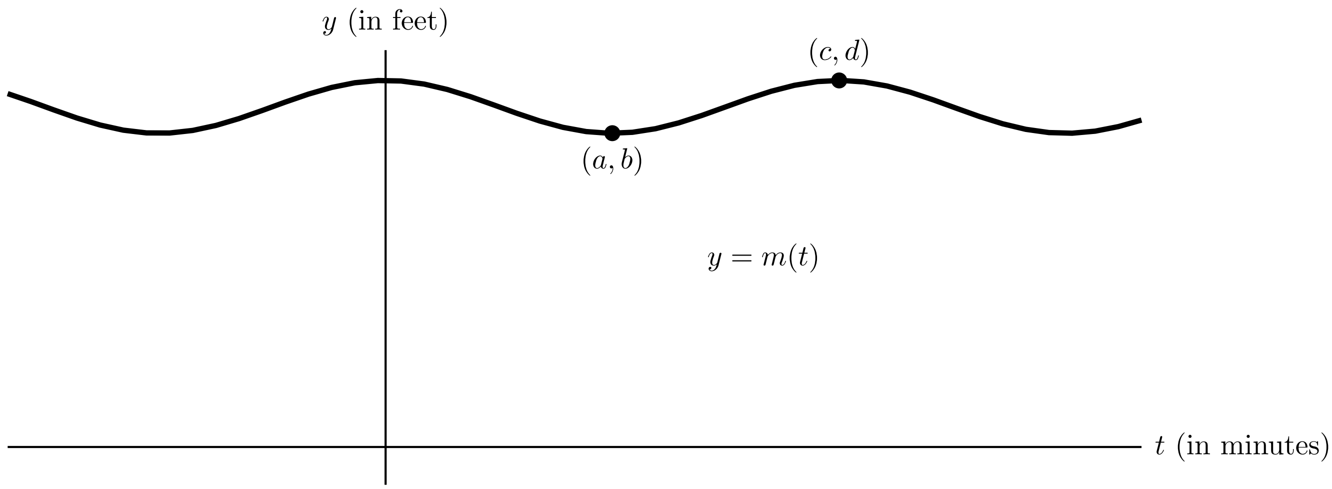
Answer: $h(x) =$ $3e^{2(x-4)} - 5$

e. [3 points] Find the *exact* value of t if $5e^t = 15(2^t)$. (*Show each step of your work carefully.*)

Solution: First, we divide both sides of the given equation by 5 to obtain $e^t = 3(2^t)$. Using logarithms, we then find $\ln(e^t) = \ln(3(2^t))$ so $t = \ln(3) + \ln(2^t) = \ln 3 + t \ln 2$. Collecting the terms involving t, we find $t - t \ln 2 = \ln 3$. We can then pull out the common factor t on the left side to find $t(1 - \ln 2) = \ln 3$ and then divide to obtain the solution $t = \dfrac{\ln 3}{1 - \ln 2}$.

Answer: $t = \dfrac{\ln 3}{1 - \ln 2}$

4. [7 points] Big Ben is the third-largest free standing clock tower in the world. It has a clock on each of its four sides. The center of each clock face is 180 feet above the ground, and the minute hand of each clock is 14 feet long. Let $m(t)$ be the height above ground, measured in feet, of the tip of a minute hand t minutes after midnight.
A portion of the graph of $y = m(t)$ is shown below.



a. Use the information provided in the description above to find the values of the constants a, b, c, and d shown in the graph.

$a =$ 30 $\quad b =$ 166 $\quad c =$ 60 $\quad d =$ 194

b. Find the period, amplitude, and midline of the graph of $y = m(t)$ and find a formula for $m(t)$. (*Include units for the period and amplitude.*)

period: 60 **minutes**

amplitude: 14 **feet**

midline: $y =$ 180

formula: $m(t) = 14\cos\left(\frac{2\pi}{60}t\right) + 180$

5. [6 points] Let g be the function defined by

$$g(x) = \frac{10(x-1)(x-2)}{(2x+1)(x^2+2x-1)}.$$

Find all zeros, y-intercepts, and horizontal and vertical asymptotes of the graph of $y = g(x)$. If appropriate, write "NONE" in the answer blank provided.
(*Show your work and write your answers in exact form.*)

Solution: Note that the numerator and denominator have no common factors.

The zeros of $g(x)$ are the zeros of $10(x-1)(x-2)$ which are $x = 1$ and $x = 2$.

zero(s): $x = 1$ **and** $x = 2$

The y-intercept of $g(x)$ is $g(0) = \dfrac{10(0-1)(0-2)}{(2(0)+1)(0^2+2(0)-1)} = \dfrac{20}{-1} = -20.$

y-intercept(s): -20

As $x \to \pm\infty$ the numerator and denominator of $g(x)$ behave like their leading terms, so as $x \to \pm\infty$, the rational function $g(x)$ behaves like $\frac{10x^2}{2x^3} = \frac{5}{x}$ which approaches 0 as $x \to \pm\infty$. Therefore, $y = 0$ is a horizontal asymptote of the graph of $y = g(x)$.

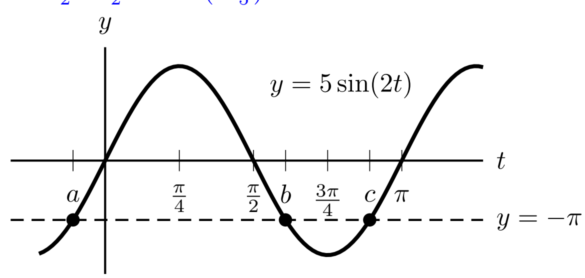
horizontal asymptote(s): $y = 0$

The vertical asymptotes of $g(x)$ are determined by the zeros of the denominator, i.e. the solutions to $(2x+1)(x^2+2x-1) = 0$ which are the solution of $2x+1 = 0$ along with the solutions of $x^2+2x+1 = 0$. The solution of $2x+1 = 0$ is $x = -1/2$ and by applying the quadratic formula, we see that the solutions of $x^2+2x-1 = 0$ are $x = \frac{-2\pm\sqrt{2^2-4(1)(-1)}}{2(1)} = \frac{-2\pm\sqrt{8}}{2} = -1 \pm \sqrt{2}$. So the graph of $y = g(x)$ has three vertical asymptotes: $x = -1/2$, $x = -1+\sqrt{2}$ and $x = -1-\sqrt{2}$.

vertical asymptote(s): $x = -1/2$, $x = -1+\sqrt{2}$, **and** $x = -1-\sqrt{2}$

6. [5 points] Find all solutions to the equation $5\sin(2t) = -\pi$ for $0 \le t \le \pi$.
(*Show your work clearly and give your final answer(s) in exact form.*)

Solution:
We first find one solution to the given equation. Since $5\sin(2t) = -\pi$ we have $\sin(2t) = -\pi/5$. Then one solution is given by $2t = \arcsin(-\pi/5)$, i.e. $t = \frac{1}{2}\arcsin\left(-\frac{\pi}{5}\right)$. To find all solutions in the domain $[0, \pi]$, we consider the graph of $y = 5\sin(2t)$ (shown to the right). The graph intersects the line $y = -\pi$ at two points for $0 \le t \le \pi$. These two points of intersection give the solutions b and c marked on the t–axis. These are the solutions we are looking for. The solution we found algebraically is marked a on the t-axis.
From the graph, we see that c is exactly one period after a, that is $c = \pi + a$.
Using symmetry, we find that $b = \frac{\pi}{2} - a$. (There are many ways to see this. For example, note that $\frac{3\pi}{4}$ is halfway between b and c. Therefore $\frac{b+c}{2} = \frac{3\pi}{4}$ so $b = \frac{3\pi}{2} - c = \frac{3\pi}{2} - (\pi + a) = \frac{\pi}{2} - a$.) Hence the two solutions to the equation $5\sin(2t) = -\pi$ for $0 \le t \le \pi$ are $t = \pi + \frac{1}{2}\arcsin\left(-\frac{\pi}{5}\right)$ and $t = \frac{\pi}{2} - \frac{1}{2}\arcsin\left(-\frac{\pi}{5}\right)$.



Answer(s): $t = \pi + \dfrac{1}{2}\arcsin\left(-\dfrac{\pi}{5}\right)$ **and** $t = \dfrac{\pi}{2} - \dfrac{1}{2}\arcsin\left(-\dfrac{\pi}{5}\right)$

7. [11 points] In honor of a favorite video game, a group of students decides to build a huge slingshot on the Diag from which they will launch a variety of large toy stuffed animals.

The first "passenger" is a large stuffed panda. The height of the panda above the ground (measured in feet) t seconds after it is launched from the slingshot is $P(t) = -16t^2 + 48t + 8$.

a. [3 points] How long is the flying stuffed panda in the air before it lands back on the ground? (*Show your work and give your answer in exact form or rounded to three decimal places.*)

Solution: The panda lands on the ground when $P(t) = 0$, i.e. when $-16t^2 + 48t + 8 = 0$. We first divide both sides of this equation by -8 to simplify it to $2t^2 - 6t - 1 = 0$.
Applying the quadratic formula* we find $t = \dfrac{6 \pm \sqrt{(-6)^2 - 4(2)(-1)}}{2(2)} = \dfrac{6 \pm \sqrt{44}}{4} = \dfrac{3 \pm \sqrt{11}}{2}$.
The solution that makes sense in context is the positive one, i.e. $t = \dfrac{3+\sqrt{11}}{2} \approx 3.158$. Hence, the flying stuffed panda is in the air for approximately 3.158 seconds before it lands on the ground.
(*Note that we could instead use a graphing calculator to estimate the positive zero of the function P.)

Answer: $\dfrac{3+\sqrt{11}}{2} \approx 3.158$ **seconds**

b. [4 points] Use the method of completing the square to rewrite the formula for $P(t)$ in vertex form. (*Carefully show your work step-by-step.*)

Solution:

$$\begin{aligned} P(t) &= -16t^2 + 48t + 8 = -16(t^2 - 3t) + 8 \quad \text{(factor out leading coefficient)} \\ &= -16\left(t^2 - 3t + \frac{9}{4} - \frac{9}{4}\right) + 8 \quad \text{(add and subtract } \left(-\tfrac{3}{2}\right)^2 = \tfrac{9}{4}\text{)} \\ &= -16\left(\left(t - \frac{3}{2}\right)^2 - \frac{9}{4}\right) + 8 \quad \text{(rewrite perfect square)} \\ &= -16\left(t - \frac{3}{2}\right)^2 + 36 + 8 = -16\left(t - \frac{3}{2}\right)^2 + 44 \quad \text{(distribute and simplify)} \end{aligned}$$

Answer: $P(t) = -16\left(t - \frac{3}{2}\right)^2 + 44$

c. [2 points] After how many seconds does the flying stuffed panda reach its maximum height above the ground? What is that maximum height?

Solution: We see from part (b) that the vertex of $y = P(t)$ is the point $(1.5, 44)$. Since the leading coefficient if $P(t)$ is negative, the graph of P is a parabola that opens downward, so P achieves a maximum at this vertex.

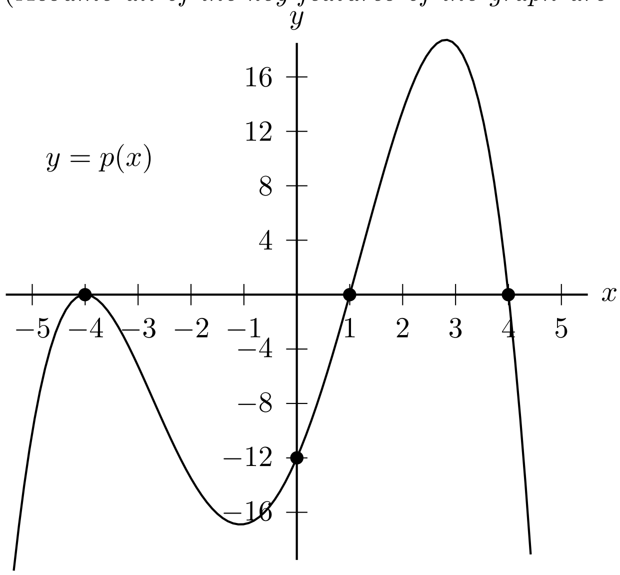
After 1.5 seconds, the panda reaches its maximum height of 44 feet.

d. [2 points] In the context of this problem, what are the domain and range of $P(t)$? (*Use either inequalities or interval notation to give your answers.*)

Solution: The answers to parts (a) and (c) above give us the domain and range.

Domain: $[0, (3+\sqrt{11})/2]$ **Range:** $[0, 44]$

8. [5 points] A portion of the graph of a polynomial function p is shown below.
Find a possible formula for $p(x)$.
(*Assume all of the key features of the graph are shown.*)



Solution: The zeros of $p(x)$ are $x = -4$, $x = 1$, and $x = 4$. Note that $x = -4$ is a double (or other positive even power) zero while $x = 1$ and $x = 4$ appear to be simple zeros. So a possible formula for $p(x)$ is $p(x) = a(x+4)^2(x-1)(x-4)$ for some (negative) constant a. Using the y-intercept, we see that $-12 = a(0+4)^2(0-1)(0-4)$, so $a = -12/64 = -3/16$.

Answer: $p(x) =$ $-\frac{3}{16}(x+4)^2(x-1)(x-4)$

9. [4 points] Suppose g is a power function such that $g(1) = 4$ and $g(10) = 1$.
Find a formula for $g(x)$. (*Any numbers in your formula should be in exact form.*)

Solution: Since g is a power function there are constants k and p so that a formula for $g(x)$ is $g(x) = kx^p$. Using the given data, we have $4 = k(1^p)$ so $4 = k$. Then $1 = k(10^p) = 4(10^p)$. Solving for p we have

$$\begin{aligned} 4(10^p) &= 1 \\ 10^p &= \frac{1}{4} \\ \log(10^p) &= \log(1/4) \\ p &= -\log 4 \end{aligned}$$

Hence a formula for $g(x)$ is $g(x) = 4x^{-\log 4}$ (which can also be written as $\frac{4}{x^{\log 4}}$ or $4x^{\log 0.25}$).

Check: $g(1) = 4(1^{-\log 4}) = 4(1) = 4$ and $g(10) = 4(10^{-\log 4}) = \frac{4}{10^{\log 4}} = \frac{4}{4} = 1$ as required.

Answer: $g(x) =$ $4x^{-\log 4}$

10. [11 points] An effective cleaning solution can be made by mixing vinegar and water. Starting with 2 liters of a solution that is one-half water and one-half vinegar, v liters of vineger are added to the solution. Let $C = g(v)$ be the concentration of vinegar in the resulting solution.

That is, $g(v) = \dfrac{\text{Total volume of vinegar}}{\text{Total volume of solution}}$ after v liters of vinegar are added.

a. [2 points] Find a formula for $g(v)$.

Solution: Since the initial two liters of solution is one-half water and one-half vinegar, there is initially one liter of vinegar and one liter of water. When v liters of vinegar are added, the resulting solution has $1 + v$ liters of vinegar and still only one liter of water, so the total volume of solution is $2 + v$ liters. Hence a formula for $g(v)$ is $g(v) = \dfrac{1+v}{2+v}$.

Answer: $g(v) = \dfrac{1+v}{2+v}$

b. [2 points] Describe, in the context of this problem, the behavior of $g(v)$ as $v \to \infty$.

Solution: As $v \to \infty$, the output $g(v)$ approaches 1 (since $\frac{1+v}{2+v}$ behaves like $\frac{v}{v}$ as $v \to \infty$). The graph of $y = g(v)$ has a horizontal asymptote of $y = 1$. In the context of this problem, this means that as more vinegar is added to the solution, the concentration of vinegar in the solution gets closer and closer to 100%. (Since no water is removed, the concentration never actually reaches 100%, but it gets arbitrarily close to 100%.)

c. [4 points] Find a formula for $g^{-1}(C)$.

Solution: We must solve for v in the equation $C = \dfrac{1+v}{2+v}$.

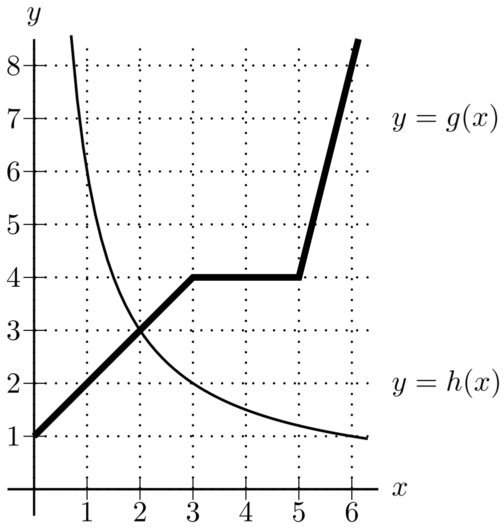
$$\begin{aligned}
C &= \frac{1+v}{2+v} \\
C(2+v) &= 1+v \\
2C + Cv &= 1+v \\
Cv - v &= 1-2C \\
v(C-1) &= 1-2C \\
v &= \frac{1-2C}{C-1}
\end{aligned}$$

Answer: $g^{-1}(C) = \dfrac{1-2C}{C-1}$

d. [3 points] Find and interpret, in the context of this problem, $g^{-1}(0.75)$.

Solution: Using the formula we found in part (c), we have $g^{-1}(0.75) = \dfrac{1-2(0.75)}{0.75-1} = \dfrac{-0.5}{-0.25} = 2$. In context, this means that in order to achieve a concentration of 75% vinegar, a total of 2 liters of vinegar must be added to the original solution.

11. [9 points] The graphs of functions g and h are shown below.



a. [3 points] *Determine whether each of the following statements is* TRUE *or* FALSE.

(i) The function g is invertible on the domain $[1, 6]$.

True $\boxed{\text{False}}$

(ii) The function h is invertible on the domain $[1, 6]$.

$\boxed{\text{True}}$ False

(iii) The function defined by $g(x) - h(x)$ is an increasing function on the domain $[1, 6]$.

$\boxed{\text{True}}$ False

b. [2 points] Evaluate $g(h(3))$ and $h(3)g(3)$.

Solution: $g(h(3)) = g(2) = 3$ and $h(3)g(3) = 2 \cdot 4 = 8$.

Answers: $g(h(3)) =$ 3 $h(3)g(3) =$ 8

Some values for an invertible function f are given in the table below.
Use the table together with the graphs of g and h above to answer the questions that follow.

x	0	1	3	5	6
$f(x)$	1	3	4	6	8

c. [2 points] Evaluate $f^{-1}(g(2))$.

Solution: $f^{-1}(g(2)) = f^{-1}(3) = 1$.

Answer: $f^{-1}(g(2)) =$ 1

d. [2 points] If j is the function defined by $j(x) = 2f(x + 1)$, evaluate $j(4)$.

Solution: $j(4) = 2f(4 + 1) = 2f(5) = 2(6) = 12$.

Answer: $j(4) =$ 12

12. [8 points] "*Timely Time*" is a local company that builds and sells clocks and watches. Let $C(q)$ be the cost (in dollars) for *Timely Time* to produce q wall clocks

a. [2 points] Write an equation that expresses the statement

"The cost of producing k clocks is m dollars."

Answer: $C(k) = m$

b. [2 points] Write an equation that expresses the fact that doubling the quantity of clocks produced increases *Timely Time's* production costs by 80%.

Answer: $C(2q) = 1.8C(q)$

Let $w(d)$ be the number of watches that can be produced by *Timely Time* for a cost of d dollars. Assume that w is an invertible function.

c. [2 points] Express the total cost for *Timely Time* to produce 15 clocks and 7 watches in terms of C and w.

Answer: $C(15) + w^{-1}(7)$

d. [2 points] Suppose that $w(C(q)) > q$ for all values of q in the domain of $w(C(q))$. Give a practical interpretation of the inequality $w(C(q)) > q$ in the context of this problem.

Solution: $C(q)$ is *Timely Time*'s cost for producing q clocks. So $w(C(q))$ is the number of watches that can be produced by *Timely Time* for the amount that it costs to produce q clocks. Since $w(C(q)) > q$, *Timely Time* can produce more watches than clocks for the same cost. (It costs less for *Timely Time* to produce watches than to produce clocks.)