Math 105 — First Midterm

February 11, 2014

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	12	
3	12	
4	9	
5	10	
6	11	
7	8	
8	6	
9	7	
10	14	
Total	100	

1. [11 points] Consider the following functions described below:

a. [7 points] Water is added into an empty tank at a rate of 9 gallons per minute until it is full. Once the tank is full, the water is shut off. The tank is 5 ft tall and has capacity to store 180 gallons of water. During the time the water is entering the tank, let $H(t)$ be the depth of water (in ft) in the tank t minutes after it starts to be filled with water.

i) [1 points] How long does it take for the tank to be filled? ____________________

Solution: $\dfrac{180}{9} = 20$ minutes.

ii) [4 points] Given that the function $H(t)$ is only defined during the time that water is being added into the tank, find the domain and range of the function $H(t)$? Write your answers in interval notation or with inequalities.

Solution: Domain: $[0, 20]$ or $0 \le t \le 20$, Range: $[0, 5]$ or $0 \le H(t) \le 5$.

iii) [2 points] Is the function $H(t)$ increasing, decreasing or neither, during the time that water is being poured into the tank? Circle your answer.

Solution: **INCREASING** Decreasing Neither

b. [4 points] As part of an experiment, bacteria is deposited in a piece of raw meat. At first, the amount of bacteria grows slowly, but its rate of growth continues to increase. Let $B(t)$ be the amount of bacteria at time t (in hours). Which of the listed attributes could be true for the function $B(t)$ on its entire domain? Circle your answer.

Solution:

INCREASING Decreasing Neither increasing or decreasing

CONCAVE UP Concave down Neither concave up or concave down

2. [12 points] A company manufactures helmets for skiing. Each helmet is sold at a price of 60 dollars.

a. [2 points] Let $F(h)$ be the revenue (in dollars), the total amount of money that the company receives, from selling h helmets in one month. Find a formula for $F(h)$.

Solution: $F(h) = 60h$

The cost of producing 150 helmets in one month is 7,250 dollars, and when the company increases the production to 275 helmets in one month, the company spends 11,800 dollars producing them.

b. [4 points] Let $G(h)$ be the total cost, in dollars, of producing h helmets in one month. Assuming that $G(h)$ is a linear function, find a formula for $G(h)$. Show all your work.

Solution: $m = \dfrac{11800 - 7250}{275 - 150} = 36.4$

Point slope formula:

$G(h) - 11800 = 36.4(h - 275)$ yields

$G(h) = 11800 + 36.4(h - 275) = 1790 + 36.4h$

Slope-intercept formula:

Since $G(h)$ linear, then $G(h) = 36.4h + b$, using one of the points in the graph $11800 = 36.4(275) + b$. Hence $b = 1790$.

Therefore $G(h) = 36.4h + 1790$.

c. [3 points] Find and give a practical interpretation of the slope of $G(h)$.

Solution: Slope$= 36.4$

Practical interpretation of the slope: The total cost of producing helmets increases by 36.40 dollars per additional helmet produced.

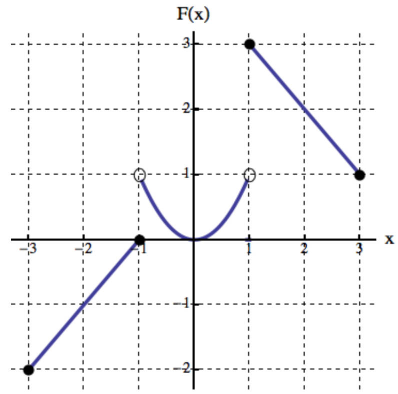
d. [3 points] The company is considering changing the selling price of each helmet. If the company produces 500 helmets in a month and sells all of them, what should be the new price for each helmet in order for the company to break even (i.e. the price of each helmet at which the company do not lose or gain any money)? Show all your work. Your answer should include cents.

Solution: The cost of producing 500 helmets is $G(500) = 36.4(500) + 1790 = 19990$. The revenue of selling 500 helmets at price p is $F(500) = 500p$. To break even $G(500) = F(500)$ $(19990 = 500p)$, hence $p = 39.98$.

New price$= 39.98$ dollars.

3. [12 points] Consider the functions $F(x)$ and $G(x)$ given below. Assume the function $G(x)$ is invertible.

x	-1	0	1	2	3
$G(x)$	-3	-1	0.3	1	2



Compute the following quantities. Write UNDEFINED if the quantity can't be computed with the information provided.

a. [2 points] $G(1)$=________________ $G^{-1}(1)$=________________

Solution: $G(1) = 0.3$ $G^{-1}(1) = 2$

b. [2 points] $F(G(2))$=________________ $F(F(-3))$=________________

Solution: $F(G(2)) = F(1) = 3$ $F(F(-3)) = F(-2) = -1$.

c. [3 points] Solve the following equations:

$F(a) = 0$ a=________________ $G(b) = -3$ b=________________.

Solution:
$F(a) = 0$ $a = -1, 0$ $G(b) = -3$ $b = -1$.

d. [2 points] If $G(F(x)) = 2$, then x=________________.

Solution: If $G(F(x)) = 2$, then $F(x) = 3$ and $x = 1$.

e. [3 points] Let $R(x)$ be defined as follows

$$R(x) = \begin{cases} 4x^2 & x \le 1 \\ 1 + 2x & x > 1. \end{cases}$$

For $h > 0$, find an expression for $R(1+h) - R(1)$ only in terms of h. No need to simplify.
$R(1+h) - R(1)$=________________________________

Solution: $R(1+h) - R(1) = 1 + 2(1+h) - 4$.

4. [9 points]

a. [4 points] A population of frogs lives in the forest. In 2000, there are 2500 frogs in the forest. The frog's population decreases at a rate of 6.2% per year. Let $f(t)$ be the number of frogs in the forest t years after 2000.

i) [3 points] Find a formula for $f(t)$, assuming the decay rate of the population of frogs continues at the same percent rate per year.

Solution: $f(t) = 2500(.938)^t$

ii) [1 points] How many frogs are in the forest in 2008? ________________

Solution: $f(8) = 2500(.938)^8 \approx 1498.18$ (or 1498 frogs).

b. [5 points] In the same forest there is a population of 1400 birds on the first day of October. Winter is arriving, and the birds are migrating to a warmer place. Every day, 25 birds leave the forest. Let $B = b(d)$ be the number of birds left in the forest, d days after October 1st.

i) [2 points] Find a formula for $b(d)$.

Solution: $b(d) = 1400 - 25d$.

ii) [3 points] Find and give a practical interpretation of the horizontal intercept of the graph of $B = b(d)$.

Horizontal intercept= ________________

Practical interpretation:

Solution: Horizontal intercept: $b(d) = 0$, then $d = \frac{1400}{25} = 56$.

Practical interpretation: It takes 56 days after Oct 1st for all the birds to leave the forest.

5. [10 points] A rocket is launched from the ground. The rocket consumes fuel as it moves away from the ground. As the rocket increases in altitude, the atmospheric pressure decays. Consider the following functions:

i) $H(t)$ is the height of the rocket above the ground, in kilometers, t seconds after being launched.

ii) $P(h)$ is the atmospheric pressure, in inches of mercury, h kilometers above the ground.

iii) $G(t)$ is the total amount of fuel burned by the rocket, in gallons, t seconds after it was launched.

Assume that $H(t)$ and $P(h)$ and $G(t)$ are invertible functions.

a. [8 points] For each of the sentences below, fill in the blank with one expression from the list of possible answers given below that makes the statement true"

Possible answers:

$P^{-1}(H^{-1}(15))$	15	$P(H(15))$	$H^{-1}(P^{-1}(15))$
$G^{-1}(15)$	$H^{-1}(15)$	$H(P(15))$	$H(15)$
$P(15)$	$P^{-1}(15)$	$G(15)$	$\frac{1}{15}$

Solution:

i) After $H^{-1}(15)$ seconds, the rocket's height above ground is **15** kilometers.

ii) The rocket is $\mathbf{H(15)}$ kilometers above the ground after 15 seconds.

iii) The atmospheric pressure around the rocket 15 seconds after the rocket was launched is $\mathbf{P(H(15))}$ inches of mercury.

iv) It takes $\mathbf{G^{-1}(15)}$ seconds for the rocket to burn 15 gallons of fuel.

b. [2 points] Suppose that the constants a and b satisfy $G(H^{-1}(a)) = b$. What are the units of a and b ?

Solution: Units of a: **kilometers** Units of b: **gallons**.

6. [11 points] Sixty liters of chlorine were accidentally spilled into a lagoon. The cost C (in millions of dollars) of removing y liters of chlorine from the water in the lagoon is given by the function

$$C(y) = \frac{y}{60 - y}.$$

a. [2 points] What is the cost of removing 10 liters of chlorine from the lagoon? Include units.

Solution: $C(10) = \frac{1}{5}$ million dollars.

b. [4 points] Compute the average rate of change of C for y between 25 and 40. Include units.

Solution: Average rate of change of C of $25 \leq y \leq 40 = \dfrac{2 - \frac{5}{7}}{40 - 25} = \dfrac{3}{35} \approx 0.085$ millions of dollars per liter.

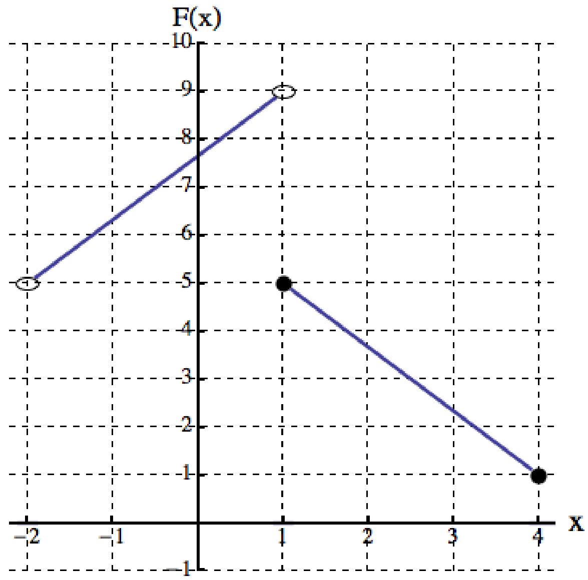
c. [3 points] How many liters of chlorine can be removed from the lagoon if you invest 3 million dollars cleaning the lagoon? Show all your work.

Solution: If $C(y) = 3$, then $\frac{y}{60-y} = 3$. Hence $y = 180 - 3y$ and $y = 45$ liters.

d. [2 points] What is the domain of the cost function $C(y)$ in the context of this problem? Use inequalities or interval notation.

Solution: Domain$=[0, 60)$ or $0 \leq y < 60$.

7. [8 points] The graph of the function $F(x)$ is given below:



a. [4 points] Find the domain and range of $F(x)$. Write your answer in interval notation or with inequalities.

Solution: Domain: $(-2, 4]$ or $-2 < x \leq 4$ Range: $[1, 9)$ or $1 \leq F(x) < 9$.

b. [4 points] Use a piecewise-defined function to write a formula for $F(x)$

Solution:

$$F(x) = \begin{cases} \frac{4}{3}x + \frac{23}{3} & (\text{or } 1.33x + 7.66) \quad \text{if } -2 < x < 1 \\ -\frac{4}{3}x + \frac{19}{3} & (\text{ or } -1.33x + 6.33) \quad \text{if } 1 \leq x \leq 4. \end{cases}$$

8. [6 points]

a. [4 points] The table shows some values of the function $Q(x)$.

x	0	2	4	6	8
$Q(x)$	0.4	1	1.2	0.9	0.4

For each question below, circle your answer based on the data in the table:

i) [2 pts] Could the function $Q(x)$ be increasing, decreasing or neither on the entire interval from $x = 0$ to $x = 8$? Circle "Neither" if neither is possible.

Solution: Increasing Decreasing **NEITHER**

ii) [2 pts] Could the function $Q(x)$ be concave up, concave down or neither on the entire interval from $x = 0$ to $x = 8$? Circle "Neither" if neither is possible.

Solution: Concave up **CONCAVE DOWN** Neither

b. [2 points] For what value of C is the line $y = Cx + 11$ perpendicular to the line $2y + 5x + 7 = 0$? Show all your work.

Solution: The slope of the line $2y + 5x + 7 = 0$ (or $y = -\frac{5}{2}x - \frac{7}{2}$) is $m = -\frac{5}{2}$. Since the line $y = Cx + 11$ is perpendicular to the line $2y + 5x + 7 = 0$ we have $C = -\frac{1}{m} = \dfrac{2}{5}$.

9. [7 points]

a. [3 points] The table shows some of the values of a linear function $g(x)$

x	-2	1	5
$g(x)$	-1.2	2.7	A

What is the value of A? Show all your work.

Solution: The slope of $g(x)$ is $m = \dfrac{2.7-(-1.2)}{1-(-2))} = \dfrac{3.9}{3} = 1.3$.

The slope point formula yields $g(x) - A = 1.3(x-5)$.
Using the point $(1, 2.7)$ we get $2.7 - A = 1.3(1-5)$, which yields $A = 7.9$.

b. [4 points] The table shows some of the values of a quadratic function $q(x)$

x	-1	0	3	4
$q(x)$	0	6	0	-10

Find a formula for $q(x)$. Show all your work.

Solution: From the table we can see that $x = -1$ and $x = 3$ are zeros of $q(x)$. Using the factored form formula for a quadratic, $q(x) = a(x+1)(x-3)$. To find a, we use the point $(x, q(x)) = (0, 6)$. Then $6 = a(0+1)(0-3)$, which yields $a = -2$.

Hence $q(x) = -2(x+1)(x-3)$.

10. [14 points] Let $P(t)$ be the price of a house (in thousands of dollars) t years after it was built. The function $P(t)$ is given by

$$P(t) = 5t^2 - 18t + 225.$$

a. [2 points] What is the price of the house five years after it was built? Include units.

Solution: $P(5) = 260$, then the price is 260 thousand dollars five years after it was built.

b. [3 points] Find the vertical intercept of the function $P(t)$ and provide a practical interpretation for it. Include units.

Solution: Vertical intercept$=P(0) = 225$.

Practical interpretation: The price of the new house was 225 thousand dollars.

c. [5 points] Use the method of completing the square to put the formula for $P(t)$ in vertex form. Show all your algebraic work step-by-step.

Solution:

$$\begin{aligned} P(t) &= 5t^2 - 18t + 225. \\ &= 5(t^2 - \frac{18}{5}t) + 225 \\ &= 5\left(t^2 - \frac{18}{5}t + (1.8)^2 - (1.8)^2\right) + 225 \\ &= 5\left((t-1.8)^2 - (1.8)^2\right) + 225 \\ &= 5(t-1.8)^2 - 5(1.8)^2 + 225 \\ &= 5(t-1.8)^2 + 208.8. \end{aligned}$$

Problem continued from the previous page.

Let $P(t)$ be the price of a house (in thousands of dollars) t years after it was built. The function $P(t)$ is given by

$$P(t) = 5t^2 - 18t + 225.$$

d. [2 points]

What is the highest price of the house during the first 5 years after it was built? In what year was the highest price attained?

Solution: After 5 years:
Highest price = 260 thousand dollars.

Highest price of the house when $t = 5$.

e. [2 points]

What is the lowest price of the house during the first 5 years after it was built? In what year was the lowest price attained?

Solution: The minimum of $P(t)$ is at the vertex $(1.8, 208.8)$
Lowest price= 208.8 thousand dollars $(208, 800$ dollars)

Lowest price of the house when $t = 1.8$.