Math 105 — Second Midterm

March 20, 2014

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	6	
2	11	
3	8	
4	14	
5	9	
6	12	
7	13	
8	10	
9	17	
Total	100	

1. [6 points]

a. [2 points] Some of the values of the function $V(x)$ are given in the following table:

x	-4	-2	0	2	4
$V(x)$	7	-3	0	3	-7

Given the information in the table, is it possible for the function $V(x)$ to be even or odd? Circle your answer, if both are impossible, circle Neither.

Solution: EVEN **ODD** NEITHER.

b. [2 points] Let $f(t) = \dfrac{1+t^4}{t^2-1}$. Is the function $f(t)$ even, odd or neither? Circle your answer.

Solution: **EVEN** ODD NEITHER.

c. [2 points] The function $H(x)$ is an odd function satisfying $\lim_{x\to 1^-} H(x) = \infty$. Find the value of $\lim_{x\to -1^+} H(x) =$ __________________.

Solution: $\lim_{x\to -1^+} H(x) = -\infty$

2. [11 points] Solve all the following equations algebraically. Your answers must be written in **exact** form. Show all your work to receive full credit.

a. [3 points] $10^{3\log(x)} = 7$

Solution:

$$\begin{aligned} 10^{3\log(x)} &= 7 \\ 10^{\log(x^3)} &= 7 \\ x^3 &= 7 \\ x &= 7^{\frac{1}{3}} \end{aligned}$$

b. [4 points] $\log(27y) - \log(2y+1) = 1$

Solution:

$$\begin{aligned} \log(27y) - \log(2y+1) &= 1 \\ \log\left(\frac{27y}{2y+1}\right) &= 1 \\ \frac{27y}{2y+1} &= 10 \\ 27y &= 20y + 10 \\ 7y &= 10 \\ y &= \frac{10}{7}. \end{aligned}$$

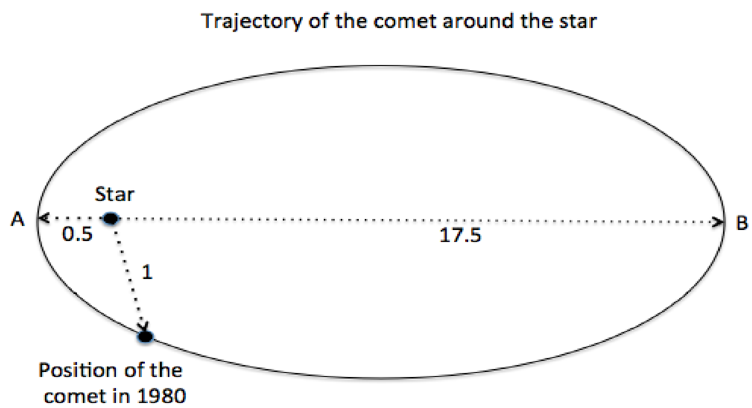
c. [4 points] $z\ln(7z+17) = 0$

Solution:

$$z = 0$$

$$\begin{aligned} \ln(7z+17) &= 0 \\ e^{\ln(7z+17)} &= e^0 \\ 7z + 17 &= 1 \\ 7z &= -16 \\ z &= -\frac{16}{7} \end{aligned}$$

3. [8 points] The trajectory of a comet around a star is shown in the figure below (figure not shown at scale).



The distance between the comet and the star is measured in astronomical units (1 astronomical unit is approximately 150 million kilometers). The following information is known about the trajectory of the comet:

i) Point A is the closest point, in the comet trajectory, to the star. The distance between the star and point A is 0.5 astronomical units.

ii) Point B is the farthest point, on the trajectory of the comet, from the star. The distance between the star and point B is 17.5 astronomic units.

iii) The comet takes 27 years to complete one trip around its trajectory.

Let $C(t)$ be the distance of the comet to the star (in astronomical units) t years after it was discovered in 1980. At the time of its discovery, the comet was 1 astronomical unit away from the star (see the figure).

a. [6 points] Find the period, amplitude and midline of the periodic function $y = C(t)$.

Solution: Since the comet takes 27 to complete one trip around its trajectory, then the period of $C(t)$ is 27 years.
The maximum distance the comet is ever from the star is 17.5 astronomical units and the minimum distance is 0.5 astronomical units. Hence the amplitude of $C(t)$ is $\frac{17.5 - .5}{2} =$ 8.5 astronomical units. The midline is given by the equation $y = \frac{17.5 + 0.5}{2} = 9$.
Period: 27 **Amplitude=8.5** and **midline: y=9**.

b. [2 points] How soon after 2014 should we expect the comet to arrive at the point in the trajectory at which it was discovered in 1980?

Solution: Twenty years after 2014 (in $1980 + 2(27) = 2034$).

4. [14 points]

a. [4 points]

i) The point P lies on a circle of radius three and it corresponds to the angle $140°$. Find the coordinates of the point P. Round your answers to the nearest 0.001.

Solution: In this problem, in order to compute the decimal approximation of the coordinates, your calculator must be in degree mode. Hence

$$P = (3\cos(140°), 3\sin 140°) = (-2.298, 1.928)$$

ii) Find angles $0° < \alpha$, $\beta < 360°$, but not equal to $140°$, such that

$\sin\alpha = \sin 140° \quad \alpha =$__________________.

$\cos\beta = \cos 140° \quad \beta =$__________________.

Solution: $\alpha = 40°$ and $\beta = 220°$.

b. [5 points] Let $0° < \theta < 45°$. For each blank space below, determine whether the number on the left is greater than, less than, or equal to the number on the right, and fill in the blank with the symbol $>$, $<$, or $=$ respectively. If there is not enough information to decide, write None in the blank space.

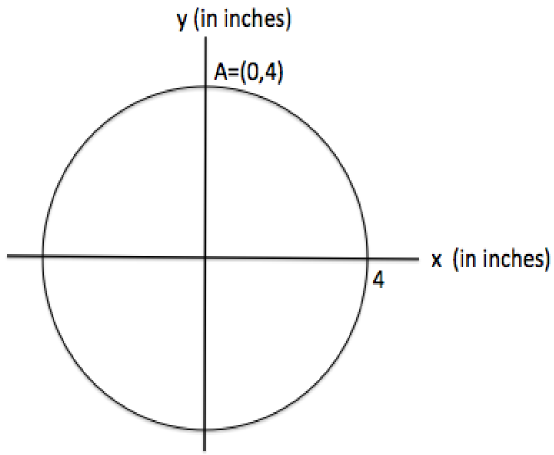
Solution:

$$\sin\theta \;>\; \sin(180° + \theta) \qquad \sin\theta \;=\; \sin(180° - \theta).$$

$$\sin\theta \;=\; \sin(720° + \theta) \qquad \cos\theta \;>\; \sin\theta.$$

$$\cos(-\theta) \;=\; \cos\theta.$$

c. [5 points] A beetle starts at the point $A = (0, 4)$ on a circle with radius of 4 inches centered at the origin. The beetle walks ten seconds at a constant speed of 0.5 inches per second around the circle in the *clockwise* direction. Find the **exact** coordinates of the final location of the beetle on the circle. Show all your work.



Solution: The beetle walks a total distance on the circle (arc length) of 5 inches. Hence the angle between the line connecting the the point with the origin and the positive y-axis is $\frac{5}{4}$ radians (using the arc length formula $s = r\theta$ with $s = 5$ and $r = 4$). At that point, the angle between the line connecting the the point with the origin and the positive x-axis is $\frac{\pi}{2} - \frac{5}{4}$. Hence the **exact** coordinates of the final location of the beetle on the circle are

$$\left(4\cos\left(\frac{\pi}{2} - \frac{5}{4}\right), 4\sin\left(\frac{\pi}{2} - \frac{5}{4}\right)\right) \textit{ or } \left(4\sin\left(\frac{5}{4}\right), 4\cos\left(\frac{5}{4}\right)\right)$$

5. [9 points] Jesse has spent the last two years learning Spanish at the Academy of Foreign Languages. He noticed that the number of words in Spanish $S(t)$ he knows is given by the formula

$$S(t) = a\log(80t + 10) - 200$$

where a is a positive constant and t is the number of years after he registered in the Academy.

a. [3 points] At the end of his first year in the Academy, Jesse knew 300 words in Spanish. What is the **exact** value of a? Show all your work.

Solution:

$$\begin{aligned} 300 &= a\log(80(1) + 10) - 200 \\ 500 &= a\log(90) \\ a &= \frac{500}{\log(90)}. \end{aligned}$$

b. [2 points] Before entering the Academy, Jesse knew some words in Spanish. How many new words has Jesse learned in the Academy during the first two years? Show all your work.

Solution:

$$\begin{aligned} a &= \frac{500}{\log(90)} \approx 255.853. \\ S(2) &= 255.853\log(170) - 200 \approx 370.668. \\ S(0) &= 255.853 - 200 = 55.853. \end{aligned}$$

Hence Jesse has learned $S(2) - S(0) \approx 370.668 - 55.853 = 314.815$, 314 new words.

c. [4 points] Assume Jesse stays in the Academy and he keeps learning new words in Spanish according to the function $S(t)$. How many years does he have to stay in the Academy in order to know 500 words in Spanish? Round your answer to the nearest 0.01 year. Show all your work.

Solution:

$$\begin{aligned} 500 &= S(t) \\ 500 &= a\log(80t + 10) - 200 \\ 700 &= a\log(80t + 10) \\ \frac{700}{a} \approx 2.735 &= \log(80t + 10) \\ 10^{2.735} &= 80t + 10 \\ 543.25 &= 80t + 10 \\ t &= 6.66 \text{ years.} \end{aligned}$$

6. [12 points] A shipment of fruit is delivered to a warehouse. The boxes containing the fruit were not properly sealed and contained fruit flies. The population of fruit flies (in thousands) in the warehouse is given by the function

$$F(t) = 12 - 10\, e^{-0.17t}$$

where t is the number of days after the fruit was delivered to the warehouse. Assume that there were no fruit flies in the warehouse before the fruit was delivered.

a. [2 points] How many fruit flies entered the warehouse when the fruit was delivered? Include units.

Solution: $F(0) = 12 - 10 = 2$. Two thousand fruit flies.

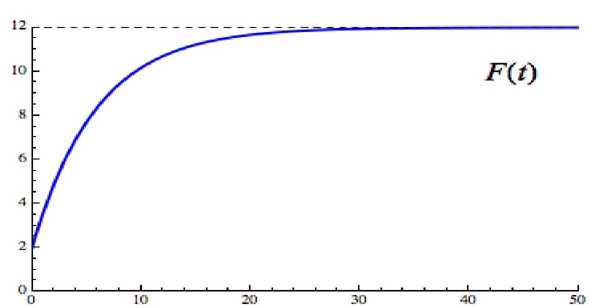
b. [4 points] How long did it take for the population of fruit flies to double after the fruit was delivered into the warehouse? Show all your work and include units.

Solution:

$$\begin{aligned} 4 &= 12 - 10\, e^{-0.17t} \\ 8 &= 10\, e^{-0.17t} \\ 0.8 &= e^{-0.17t} \\ -0.17t &= \ln(0.8) \\ t &= \frac{\ln(0.8)}{-0.17} \approx 1.312 \text{ days.} \end{aligned}$$

c. [2 points] Use your graphing calculator to find $\lim\limits_{t\to\infty} F(t)$. Include a sketch of the graph to support your answer.

Solution:



$$\lim_{t\to\infty} F(t) = 12$$

Problem continues on next page

The statement of the problem has been rewritten for your convenience:

A shipment of fruit is delivered to a warehouse. The boxes containing the fruit were not properly sealed and contained fruit flies. The population of fruit flies (in thousands) in the warehouse is given by the function

$$F(t) = 12 - 10\, e^{-0.17t}$$

where t is the number of days after the fruit was delivered to the warehouse. Assume that there were no fruit flies in the warehouse before the fruit was delivered.

d. [4 points] Five days after the fruit was delivered to the warehouse, a powerful pesticide is applied to control the population of fruit flies. The pesticide causes the population of fruit flies to decay at a continuous rate of 41% per day. Find a formula for $P(T)$, the number of fruit flies (in thousands) T days after the pesticide was applied.

Solution: After 5 days, there are $F(5) = 12 - 10\, e^{-0.17(5)} = 7.725$ thousand fruit flies. Hence $P(T) = 7.725e^{-.41T}$.

7. [13 points] A video is posted on the internet. At 2 pm, the video had 2400 views. By 5 pm, the video had 4000 views. Let $V(t)$ the number of views t hours after noon. Assume that $V(t)$ grows exponentially.

a. [5 points] Find a formula for $V(t)$. You must find this formula algebraically. All your numbers in your formula should be in **exact** form. Show all your work.

Solution: If $V(t) = ab^t$, then

$$\begin{aligned} ab^2 &= 2400 \\ ab^5 &= 4000 \end{aligned}$$

Then $b^3 = \frac{5}{3}$. Which yields $b = \left(\frac{5}{3}\right)^{\frac{1}{3}}$. Using

$$\begin{aligned} ab^2 &= 2400 \\ a\left(\frac{5}{3}\right)^{\frac{2}{3}} &= 2400 \\ a &= \frac{2400}{\left(\frac{5}{3}\right)^{\frac{2}{3}}} \end{aligned}$$

Hence $V(t) = \frac{2400}{\left(\frac{5}{3}\right)^{\frac{2}{3}}}\left(\frac{5}{3}\right)^{\frac{t}{3}} = 2400\left(\frac{5}{3}\right)^{\frac{t-2}{3}}$

b. [2 points] How many views did the video have at noon?

Solution: $V(0) = a = \frac{2400}{\left(\frac{5}{3}\right)^{\frac{2}{3}}} \approx 1707.308$, 1707 views.

c. [4 points] How long will it take for the video to have 10 thousand views? Round your answer to the nearest 0.01 hour. Show all your work.

Solution: Since $b = \left(\frac{5}{3}\right)^{\frac{1}{3}} \approx 1.185$, then $V(t) = 1707.308(1.185)^t$. In order to be 10 thousand views by time t,

$$\begin{aligned} 10000 &= 1707.308(1.185)^t \\ 5.857 &= (1.185)^t \\ \ln(5.857) &= t\ln(1.185) \\ t &= \frac{\ln(5.857)}{\ln(1.185)} \approx 10.413 \text{ hours.} \end{aligned}$$

d. [2 points] What is the continuous growth rate per hour of $V(t)$? Round your answer to the nearest 0.01%.

Solution: $1.185 = e^k$, then $k = \ln(1.185) = .16974$, then the continuous growth rate per hour of $V(t)$ is 16.97% .

8. [10 points]

a. [4 points] A corporation owns two factories that produce light bulbs. The factories are located in Ann Arbor and Detroit. On one particular day, both factories begin producing light bulbs at 6 am. Let $a(t)$ be the total amount of light bulbs that the factory in Ann Arbor produced that day, t hours after 6 am. Find a formula for the following functions in terms of transformations to the function $a(t)$.

i) Let $g(t)$ be the total amount of light bulbs produced by the factory in Ann Arbor so far that day, t hours after **9 am**.

Solution: $g(t) = a(t+3)$

ii) The factory in Detroit is larger and produces double the amount of light bulbs than the factory in Ann Arbor three times faster. Let $d(t)$ be the total amount of light bulbs produced by the factory in Detroit so far that day, t hours after 6 am.

Solution: $d(t) = 2a(3t)$

b. [6 points] A company sells bread to a small city. The company can produce $L(w)$ loaves of bread in a month with w kilograms of wheat. Let p_0 be the average amount of wheat (in kilograms) that the company uses each month and q_0 be the average amount of loaves the company produces monthly. Answer the following questions, the function L and the constants p_0 and q_0 may appear in your answers.

i) This month, the company used half the average amount of wheat for their monthly

production of bread, hence it will produce ________________ loaves of bread.

Solution: Answer: $L(\frac{1}{2}p_0)$.

ii) Find an equation expressing the following fact: If the company uses 100 kilograms more than the average amount of wheat for their monthly production of bread, then it will produce 12% more than their average monthly production of bread.

Equation: ________________________________

Solution: $L(p_0 + 100) = 1.12q_0$.

9. [17 points]

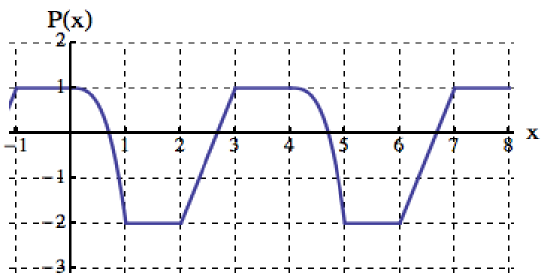
a. [2 points] Suppose the point $(2,1)$ is in the graph of $y = V(x)$. What point is in the graph of $H(x) = 9V(x-5)$?

Solution: Point: (7, 9)

b. [4 points] Suppose that $Q(x)$ has a vertical asymptote at $x = -5$ and a horizontal asymptote at $y = 2$. Find the equation(s) of the vertical and horizontal asymptotes of the function $K(x) = 3 - Q(2x-1)$.

Solution: Vertical asymptote: $x = -2$ Horizontal asymptote: $y = 1$.

c. [6 points] The graph of a periodic function $y = P(x)$ is shown below



Consider the periodic function $Q(x) = 4P(\frac{1}{2}x - 1) + 5$. Find the period, the amplitude and the midline of the functions $y = P(x)$ and $y = Q(x)$.

Solution:

i) Period of $P(x)$: 4 Period of $Q(x)$: 8.

ii) Midline of $P(x)$: $y = -0.5$ Midline of $Q(x)$: $y = 3$.

iii) Amplitude of $P(x)$: 1.5 Amplitude of $Q(x)$: 6

d. [5 points] The graph of $y = L(w)$ can be obtained from the graph of $y = e^w$ by doing the following transformations in the given order:

1. Vertical compression by a factor of $\frac{1}{3}$.
2. Horizontal stretch by a factor of 2.
3. Reflection across the y-axis.
4. Horizontal shift to the right by 5.
5. Vertical shift down by 4.

Find a formula for $L(w) =$ ______________________

Solution: $L(w) = \dfrac{1}{3}e^{-\frac{1}{2}(w-5)} - 4 = \dfrac{1}{3}e^{-\frac{1}{2}w+\frac{5}{2}} - 4$