Math 105 — Final Exam

April 28, 2014

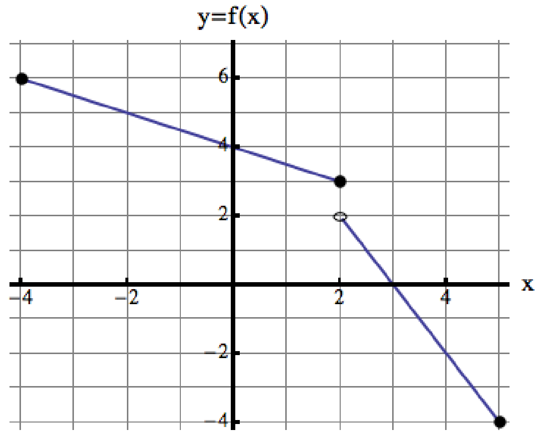
Name: EXAM SOLUTIONS

Instructor: __________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	13	
2	12	
3	8	
4	12	
5	11	
6	6	
7	8	
8	9	
9	13	
10	8	
Total	100	

1. [13 points] Consider the functions $f(x)$ and $g(x)$, where $g(x) = 2 - \frac{1}{2}x$ and the graph of $y = f(x)$ is shown below.



a. [9 points]

i) Compute the value of the following expressions. Write "Undefined" if the value of the expression is not defined or there is not enough information to be computed.

Solution: $2f(-2) + 3f(4) = 2(5) + 3(-2) = \mathbf{4}$ $\qquad f^{-1}(3) = \mathbf{2}$

$f(g(2)) = f(2 - 0.5(2)) = f(1) = \mathbf{3.5}$ $\qquad g(g^{-1}(5)) = \mathbf{5}$

ii) Find the horizontal and vertical intercepts of the function $y = f(g(x))$.

Solution:
Horizontal intercept: If $f(g(x)) = 0$ then $g(x) = 3$. Hence $2 - \frac{1}{2}x = 3$ then $x = -2$. Hence the horizontal intercept is at **(-2,0)**.

Vertical intercept: $y = f(g(0)) = f(2) = 3$. Hence the vertical intercept is at **(0,3)**.

iii) Find the average rate of change of $f(x)$ between $x = 2$ and $x = 5$. Show your work.

Solution: Average rate of change of $f(x) = \dfrac{f(5) - f(2)}{5 - 2} = \dfrac{-4 - 3}{3} = -\dfrac{7}{3}$.

b. [4 points] Find a piecewise defined formula for $f(x)$.

Solution:

$$f(x) = \begin{cases} 4 - \frac{1}{2}x & -4 \leq x \leq 2 \\ -2x + 6 & 2 < x \leq 5 \end{cases}$$

2. [12 points]

a. [2 points] The graph of an odd function $y = f(x)$ contains the point $(-2, 4)$. What other point must be in the graph of $y = f(x)$?

Solution: (2,-4)

b. [2 points] The graph of an invertible function $g(x)$ contains the point $(3, 7)$. What point must be in the graph of $y = g^{-1}(x)$?

Solution: (7,3)

c. [4 points] The function $h(x)$ is obtained by applying the following transformations to the function $y = \sqrt{1+x}$ in this exact order:

i) A vertical shift up by 5 units.
ii) A reflection about the y-axis.
iii) A horizontal compression by $\frac{1}{7}$.
iv) A horizontal shift to the left by 3 units .

Find a formula for $h(x)$.

Solution: $h(x) = 5 + \sqrt{1 - 7(x+3)} = 5 + \sqrt{-7x - 20}$.

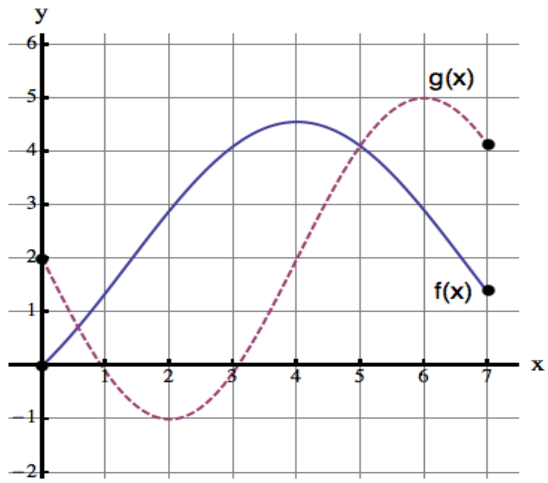
d. [4 points] Let $f(x) = (\sin(x^2) + 3)^2$ and $g(x) = x^2$. Find formulas for the functions $h(x)$ and $w(x)$ that satisfy:

i) $f(x) = g(w(x))$ $\qquad w(x) =$ ____________________.

ii) $f(x) = h(g(x))$ $\qquad h(x) =$ ____________________.

Solution: $w(x) = \sin(x^2) + 3 \qquad h(x) = (\sin(x) + 3)^2$

3. [8 points] The graphs of the functions $f(x)$ and $g(x)$ are shown below. The domain of $f(x)$ and $g(x)$ is $0 \le x \le 7$.



a. [4 points]

i) Find the range of $g(x)$. Use interval notation or inequalities in your answer.

Solution: Range=$[-1,5]$ or $-1 \le y \le 5$.

ii) For which values of $0 \le x \le 7$ is the function $g(x)$ concave down? Use interval notation or inequalities in your answer.

Solution: $g(x)$ is concave down in $[4,7]$ or $4 \le x \le 7$.

iii) For which values of $0 \le x \le 7$ is the function $g(x)$ increasing? Use interval notation or inequalities in your answer.

Solution: $g(x)$ is increasing in $[2,6]$.

b. [4 points] Define the functions:

$$D(x) = g(x) - f(x) \qquad \text{and} \qquad R(x) = \frac{g(x)}{f(x)}.$$

i) For which values of $0 \le x \le 7$ is the function $D(x)$ negative? Use interval notation or inequalities in your answer.

Solution: $D(x) < 0$ on $(0.5, 5)$ or $0.5 < x < 5$.

ii) Find the domain of the function $R(x)$. Use interval notation or inequalities in your answer.

Solution: Domain of $R(x)$: $(0,7]$ or $0 < x \le 7$.

4. [12 points]

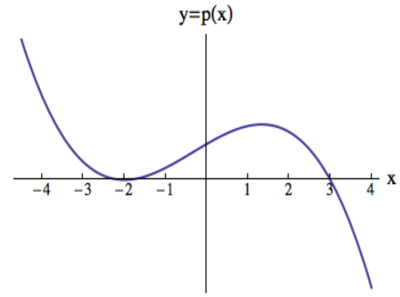
a. [4 points] The graph of a polynomial $p(x)$ is shown below. The following facts are known about $p(x)$:

i) The only zeros of $p(x)$ are $x = -2$ and $x = 3$.

ii) The degree of $p(x)$ is at most four.

iii) The point $(1, 9)$ is on the graph of $p(x)$.

Find a formula for $p(x)$.



Solution: The polynomial $p(x)$ has degree 3 given the long behavior in the graph. From the graph, we can see that $x = -2$ is a double zero. Hence $p(x) = k(x+2)^2(x-3)$. Since the point $(1, 9)$ is on the graph of $p(x)$, then $9 = k(3^2)(-2) = -18k$. Hence $k = -0.5$. Then $p(x) = -0.5(x+2)^2(x-3)$.

b. [5 points] Let

$$R(x) = \frac{(x^2+9)(10x+1)}{7x^3 - x}.$$

Find all the intercepts and all horizontal and vertical asymptotes of the graph $y = R(x)$. If appropriate, write "None" in the answer blank provided. Your answers should be in **exact form**.

Solution:

i) x-intercept(s): Set $(x^2+9)(10x+1) = 0$. Then $x^2 + 9 = 0$ (has no solutions) and $10x + 1 = 0$ implies $x = -0.1$.

ii) y-intercept(s): Since $R(0)$ is undefined, then $y = R(x)$ has **no y-intercepts**.

iii) vertical asymptote(s): Set $7x^3 - x = 0$, then $x(7x^2 - 1) = 0$ which yields $x = 0$ and $\mathbf{x} = \pm\frac{1}{\sqrt{7}}$

iv) horizontal asymptote(s): $y = \frac{10}{7}$.

c. [3 points] A law of physics states that the force F (in Newtons) exerted between two objects is inversely proportional to the square of the distance r (in meters) between them, and $F = 30$ when $r = 7$. Find a formula for F in terms of r.

Solution: Since F is inversely proportional to r^2, then $F(r) = \frac{k}{r^2}$. Using $F(7) = 30$, we get $k = (30)(7)^2 = 1,470$. Hence $F(r) = \frac{1470}{r^2}$.

5. [11 points] A package is thrown from an airplane. The height of the package (in meters) above the ground t seconds after it was thrown from the airplane is given by the function

$$H(t) = -5t^2 - 10t + 160.$$

a. [2 points] What is the height of the airplane at the time in which the package is thrown? Include units.

Height= ____________________

Solution: Height=$H(0) = 160$ meters.

b. [3 points] How many seconds does it take for the package to be 10 meters above the ground? Find your answer algebraically. Show all your work.

Solution: Solve $H(t) = 10$. In this case $-5t^2 - 10t + 160 = 10$, or $-5t^2 - 10t + 150 = 0$. Using the quadratic formula

$$t = \frac{10 \pm \sqrt{100 - 4(-5)(150)}}{-10} = \frac{10 \pm \sqrt{3100}}{-10} = -1 \pm \sqrt{31}.$$

It takes $-1 + \sqrt{31} \approx 4.56$ seconds for the package to be 10 meters above the ground.

c. [2 points] What is the range of the function $y = H(t)$ in the context of this problem? Give your answer using either interval notation or inequalities.

Solution: The values of $H(t)$ that are relevant in the context of this problem are the height of the package from the moment it is thrown from the airplane until it hits the ground, $0 \leq y \leq 160$.

d. [4 points] Another package is released from an airplane at a higher altitude. In this case, the downward velocity $V(t)$ (in meters per second) of the package t seconds after it was released is given by the function

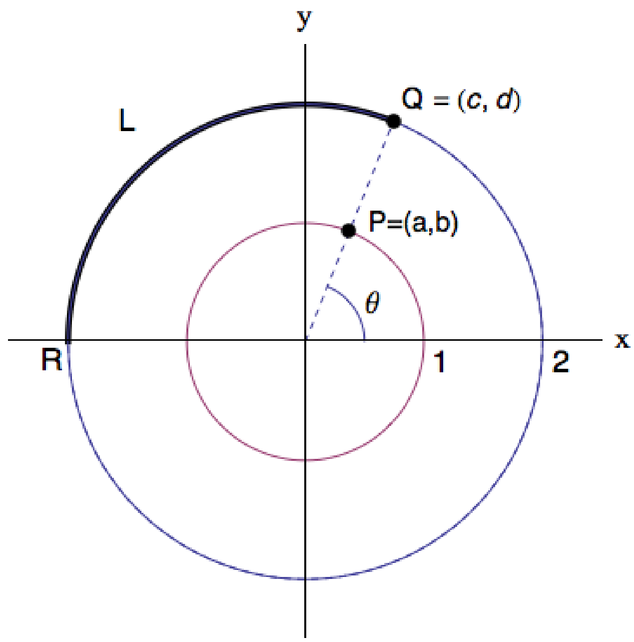
$$V(t) = 50 - 50e^{-0.2\,t}$$

How long does it take for the package to have a downward velocity of 30 meters per second? Find your answer algebraically. Show all your work step by step. Your answer must be in **exact form**.

Solution:

$$\begin{aligned} 50 - 50e^{-0.2\,t} &= 30 \\ e^{-0.2\,t} &= \frac{2}{5} \\ -0.2\,t &= \ln\left(\frac{2}{5}\right) \\ t &= -5\ln\left(\frac{2}{5}\right) \end{aligned}$$

6. [6 points] The points $P = (a, b)$ and $Q = (c, d)$ lie on the unit circle and the circle of radius 2, respectively, centered at the origin. The point P lies in the line segment between the origin and the point Q. The angle θ (measured in radians), is formed by the positive x-axis and the line between the origin and the point Q (see the figure below).



a. [2 points] Find an expression in terms of θ that computes the length L of the arc between the points Q and $R = (-2, 0)$ (see the the arc in bold in the figure above).

L=______________________________.

Solution: $L = 2(\pi - \theta)$.

b. [4 points] Find a formula for each of the quantities below **only** in terms of the constants **a** and/or **b**.

$\cos\theta =$ ____________________ $\tan\theta =$ ____________________

$c =$____________________ $\sin(\theta + \pi) =$ ____________________

Solution: $\cos\theta = a \qquad \tan\theta = \dfrac{b}{a} \qquad c = 2a \qquad \sin(\theta + \phi) = -b.$

7. [8 points] An environmental impact study has determined that most of the pollution in the air in a small town is produced by automobile exhaust. Let $P(c)$ be the level of carbon monoxide in the air (in mg per m^3) produced by c cars in this town in a day. Assume that $P(c)$ is invertible. Let $A(t)$ be the number of cars in the town, t days after January 1st, 2013 in the town.

a. [2 points] What is the practical interpretation of the vertical intercept of the function $y = A(t)$? Use a complete sentence and include units.

Solution: The vertical intercept is the number of cars in the town on January 1st, 2013.

b. [2 points] Write down a practical interpretation for the equation $P(A(2)) = 1$. Use a complete sentence and include units.

Solution: On January 3, 2013, the level of carbon monoxide in the air is 1 mg per m^3 .

c. [2 points] Write an expression for the number of cars that produce a level of carbon monoxide in the air of 10 mg per m^3 in a day in this town.

Solution: $P^{-1}(10)$.

d. [2 points] Let c_0 be the number of cars in the town during Thanksgiving day and p_0 be the average level of carbon monoxide in the air (in mg per m^3) during the year 2013. Write an equation that states the following fact:

The level of pollution in the town (in mg per m^3) during Thanksgiving day was exactly 20% higher than the average level of carbon monoxide in the air (in mg per m^3) during the year 2013.

Solution: $P(c_0) = 1.2p_0$.

8. [9 points]

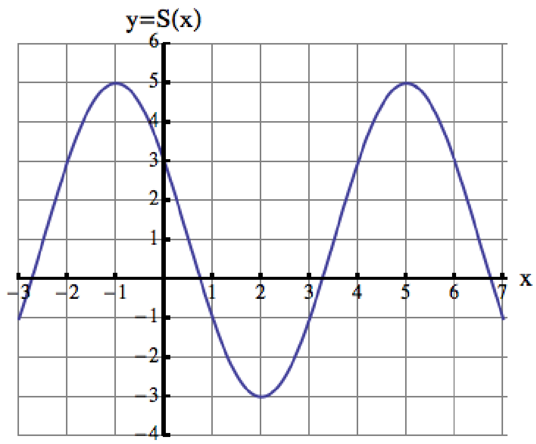
a. [4 points] A population of butterflies in a botanical garden has been found to oscillate sinusoidally. The population of butterflies reaches a maximum of 2000 butterflies followed by a minimum of 750 butterflies two months later. Let $B(t)$ be the amount of butterflies in the botanical garden at time t (in months). Find the amplitude, midline and period of the periodic function $y = B(t)$.

Amplitude: ________________.
Period: ________________.
Midline: ________________.

Solution:
Amplitude$=\dfrac{2000-(750)}{2} = 625$. Midline: $y = \dfrac{2000+(750)}{2} = 1375$

Period$=$ 4 (months).

b. [5 points] The graph of a sinusoidal function $y = S(x)$ is shown below. Find a formula for $S(x)$.



Solution: Amplitude$=\dfrac{5-(-3)}{2} = 4$. Midline: $y = \dfrac{5+(-3)}{2} = 1$ Period$=$6.

Four possible solutions (among many others):

- $y = 4\sin\left(\dfrac{2\pi}{6}(x+2.5)\right) + 1 = 4\sin\left(\dfrac{\pi}{3}(x+2.5)\right) + 1$
- $y = -4\sin\left(\dfrac{2\pi}{6}(x-0.5)\right) + 1 = -4\sin\left(\dfrac{\pi}{3}(x-0.5)\right) + 1$
- $y = 4\cos\left(\dfrac{2\pi}{6}(x+1)\right) + 1 = 4\cos\left(\dfrac{\pi}{3}(x+1)\right) + 1$
- $y = -4\cos\left(\dfrac{2\pi}{6}(x-2)\right) + 1 = -4\cos\left(\dfrac{\pi}{3}(x-2)\right) + 1$

9. [13 points] Three scientists, Laura, Emily and Patrick studied the growth of certain cells using three different lab techniques for cell growth. Suppose that they started their experiments at the same time. Let $L(t)$, $E(t)$ and $P(t)$ be the number of cells t hours after noon using Laura's, Emily's and Patrick's techniques, respectively. All the questions should be solved algebraically step by step.

a. [2 points] The amount of cells $L(t)$ increased by 315 cells every two hours. Find a formula for $L(t)$ if there were 2150 cells in Laura's experiment at 3 pm.

Solution: $L(t) = 2150 + \frac{315}{2}(t-3) = \frac{315}{2}t + \frac{3355}{2} = 157.5t + 1677.5$

b. [5 points] Emily noticed that by applying her technique, the amount of cells doubled every 4 hours.

i) Find a formula for $E(t)$ in **exact form** if there were 1500 cells at 3 pm.

Solution: $E(t) = ab^t$, then

$$\begin{aligned} 2a &= ab^4 \\ 2 &= b^4 \qquad b = 2^{\frac{1}{4}} \\ 1500 &= a2^{\frac{3}{4}} \\ a &= \frac{1500}{2^{\frac{3}{4}}} = 1500(2^{-\frac{3}{4}}). \qquad E(t) = 1500(2^{-\frac{3}{4}})2^{\frac{1}{4}t} \end{aligned}$$

ii) What is the continuous hourly growth rate of $E(t)$? Give your answer in exact form or accurate to at least three decimal places. Show all your work.

Solution: $b = e^k$, then $k = \ln b$, hence $k = \ln(2^{\frac{1}{4}}) \approx 0.173$. $k = 0.173$ or 17.3%.

c. [5 points] Patrick notices that the amount of cells in his experiment $P(t)$ is a power function. Find a formula for $P(t)$ if there are 2000 cells at 3 pm and 3000 at 5 pm. Show all your work step by step.

$P(t) =$________________________.

Solution: Since $P(t) = kt^p$, then

$$2000 = k3^p \qquad 3000 = k5^p$$
$$\frac{3000}{2000} = \frac{5^p}{3^p} = \left(\frac{5}{3}\right)^p$$
$$\frac{3}{2} = \left(\frac{5}{3}\right)^p \qquad \ln\left(\frac{3}{2}\right) = p\ln\left(\frac{5}{3}\right) \qquad p = \frac{\ln\left(\frac{3}{2}\right)}{\ln\left(\frac{5}{3}\right)} \approx 0.793.$$
$$2000 = k3^{0.793} \qquad k = \frac{2000}{3^{0.793}} \approx 836.898.$$
$$P(t) = 836.898t^{0.793}.$$

d. [1 point] Which of the three increasing functions $L(t)$, $E(t)$ and $P(t)$ grows fastest as $t \to \infty$? Circle your answer. You do not need to justify your answer.

$L(t)$ **E(t)** $P(t)$ Not enough information to conclude.

Solution: $E(t)$: Increasing exponential functions increase faster than linear and power functions.

10. [8 points]

a. [4 points] Let $y = f(x) = 3\log\left(\frac{1+2x}{x+3}\right)$. Find a formula for $f^{-1}(y)$. Show all your work carefully.

Solution:

$$\begin{aligned} y &= 3\log\left(\frac{1+2x}{x+3}\right) \\ \frac{y}{3} &= \log\left(\frac{1+2x}{x+3}\right) \\ 10^{\frac{y}{3}} &= \frac{1+2x}{x+3} \\ 10^{\frac{y}{3}}(x+3) &= 1+2x \\ 10^{\frac{y}{3}}x + 3(10^{\frac{y}{3}}) &= 1+2x \\ (10^{\frac{y}{3}} - 2)x &= 1 - 3(10^{\frac{y}{3}}) \\ f^{-1}(y) &= \frac{1-3(10^{\frac{y}{3}})}{10^{\frac{y}{3}} - 2}. \end{aligned}$$

b. [4 points] Find all solutions to $3\cos\left(\frac{t}{2}\right) + 2 = 0$ for $0 \le t \le 4\pi$ algebraically. Show all your work carefully. Your answer(s) must be in **exact form**.

Solution:

$$\begin{aligned} 3\cos\left(\frac{t}{2}\right) + 2 &= 0 \\ \cos\left(\frac{t}{2}\right) &= -\frac{2}{3}. \\ \frac{t}{2} &= \cos^{-1}\left(-\frac{2}{3}\right). \end{aligned}$$

$$t_1 = 2\cos^{-1}\left(-\frac{2}{3}\right), \quad t_2 = 4\pi - 2\cos^{-1}\left(-\frac{2}{3}\right)$$