Math 105 — First Midterm

February 10, 2015

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	8	
2	6	
3	14	
4	10	
5	12	
6	17	
7	11	
8	10	
9	12	
Total	100	

1. [8 points] Indicate if each of the following statements are true or false by circling the correct answer. No justification is required.

a. [2 points] For any function f, $f(x+3) = f(x) + f(3)$.

True $\boxed{\text{False}}$

b. [2 points] The function $k(w)$ shown in the table below could be linear.

w	2	4	7
$k(w)$	-2	1	4

True $\boxed{\text{False}}$

c. [2 points] Let the function $g(x)$ be the inverse of $h(x)$. If $h(3) = 4$, then $h(g(4)) = 4$.

$\boxed{\text{True}}$ False

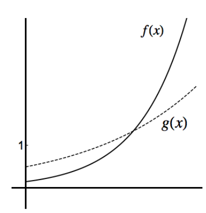
d. [2 points] According to the following table, Z could be a function of Y.

Y	2	3	3.7	4.5	5.2
Z	-2	1.5	3.4	2.6	1.5

$\boxed{\text{True}}$ False

2. [6 points]

a. [4 points] Consider the exponential functions $f(x) = ab^x$ and $g(x) = cd^x$, where a, b, c and d are positive constants. The graphs of $f(x)$ (in solid line) and $g(x)$ (in dashed line) are shown below.



Determine which of the following inequalities must be true. Circle all that apply.

Solution:

$b < d$ $\boxed{d < b}$ $\boxed{a < c}$ $c < a$ $\boxed{c < b}$ $b < c$

b. [2 points] Find the value of the constant m if the lines $2x + 4y = 5$ and $mx - 3y = 1$ are perpendicular.

Solution: The slope of the first line is $m_1 = -0.5$ and the second line $m_2 = \dfrac{m}{3}$. The lines are perpendicular if $m_1 m_2 = -1$. Then $m = 6$.

3. [14 points] Consider the functions $H(x)$, $G(x)$ and $M(x)$

x	-1	0	1	2	3
$H(x)$	2	3	-1	0	1

$$G(x) = \begin{cases} x-2 & -1 < x < 1 \\ 0 & x = 1 \\ x^3 & 1 < x \leq 3 \end{cases} \qquad M(x) = \frac{1-x}{2x}$$

Assume that the function H has an inverse.

a. [8 points] Find the value of the following mathematical expressions. If the expression is undefined, write UNDEFINED.

Solution:

$$G(1) = 0 \qquad G(H(1)) = G(-1) \quad \text{UNDEFINED}$$

$$H^{-1}(2) = -1 \qquad H(3G(0)+5) = H(3(-2)+5) = H(-1) = 2$$

$$(M(2))^{-1} = \frac{2(2)}{1-2} = -4$$

b. [3 points] Solve the equation $H(M(x)) = 0$. Show all your algebraic work.

Solution: The output of the function H is equal to zero when the input is equal to two. Then $M(x) = 2$. In this case we have

$$\frac{1-x}{2x} = 2, \quad 1 - x = 4x \quad \text{then} \quad x = \frac{1}{5}.$$

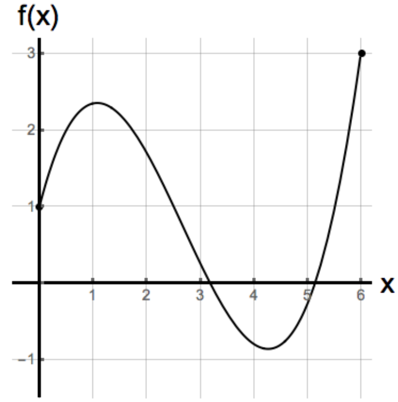
c. [3 points] What is the average rate of change of $G(x)$ for $-\frac{1}{2} \leq x \leq 3$. Show all your work.

Solution:

$$\frac{\Delta G}{\Delta x} = \frac{G(3) - G(-\frac{1}{2})}{3.5} = \frac{27 - (-2.5)}{3.5} = \frac{29.5}{3.5} = 8.428.$$

4. [10 points]

a. [4 points] Consider the function $f(x)$ whose graph is shown below.



Use the graph to answer the following questions. Your answers below should be written using interval notation or inequalities.

Solution:

i) For which values of x is the function $f(x)$ decreasing? $1.1 < x < 4.2$.

ii) For which values of x is the function $f(x)$ concave up? $2.5 < x < 6$.

b. [6 points] Determine which of the listed attributes could be true for the following functions on the entire domain given. Circle all the attributes that could be true and if none of the listed attributes can be true, circle "NONE OF THESE"

i) Some of the values of the function $g(x)$ are shown in the table below

x	2	4	6	8
$g(x)$	1000	10	2	1

then $g(x)$ could be:

Solution:

INCREASING $\boxed{\text{DECREASING}}$ EXPONENTIAL

$\boxed{\text{CONCAVE UP}}$ CONCAVE DOWN NONE OF THESE

ii) On a hot summer day, Pete buys ice cream. He forgets the ice cream in his car where the ice cream starts to warm up very rapidly at first, but then it warms up more slowly as its temperature gets closer to the car's temperature. Let $h(t)$ be the temperature of the ice cream t minutes after it was left in the car.
Then $h(t)$ could be:

Solution:

$\boxed{\text{INCREASING}}$ DECREASING LINEAR

CONCAVE UP $\boxed{\text{CONCAVE DOWN}}$ NONE OF THESE

5. [12 points] A coffee shop owner buys coffee from company A or company B. Let $A(c)$ and $B(c)$ be the cost (in dollars) of buying c pounds of coffee from company A and company B respectively. The formulas for the cost functions are given below

$$A(c) = 15 + 8.25c \qquad \text{and} \qquad B(c) = 22 + 7.85c.$$

a. [3 points] What is the practical interpretation of the slope of $A(c)$?

Solution: The cost in dollars of buying an additional pound of coffee from company A.

In the following questions, you must find all your answers ***algebraically***. Show all your work. Your answers must be accurate up to the first two decimals.

b. [2 points] How many pounds of coffee do you need to buy in order for the cost of the coffee to be the same if you buy it either from company A or company B?

Solution: We need to find the number c of lbs of coffee such that

$$\begin{aligned} 15 + 8.25c &= 22 + 7.85c. \\ 0.40c &= 7 \\ c &= \frac{7}{0.40} = 17.5 \text{ lbs of coffee.} \end{aligned}$$

c. [2 points] If the coffee shop owner wants to buy 1000 dollars worth of coffee from company A, how many pounds of coffee can he afford?

Solution:

$$\begin{aligned} 15 + 8.25c &= 1000. \\ 8.25c &= 985. \\ c &= \frac{985}{8.25} = 119.32 \text{ lbs of coffee.} \end{aligned}$$

d. [5 points] Suppose that the coffee shop owner wants to buy 500 dollars worth of coffee, but he wants to buy 50 percent more coffee from company A than from company B. How many pounds of coffee does he need to buy from company B?

Solution: Let a and b be the number of lbs of coffee that he will buy from company A and B respectively. If he wants to buy 50 percent more coffee from company A than from company B, then $a = 1.5b$. If he spends 500 dlls in coffee, then $A(a) + B(b) = 500$. Therefore

$$\begin{aligned} A(a) + B(b) &= (15 + 8.25a) + (22 + 7.85b) = 500. \\ \text{using } a = 1.5b \qquad & 15 + 8.25(1.5b) + (22 + 7.85b) = 500. \\ & 15 + 12.375b + 22 + 7.85b = 500 \\ & 37 + 20.225b = 500 \\ & 20.225b = 463 \qquad b = 22.89 \text{ lbs of coffee from company B.} \end{aligned}$$

6. [17 points] Luis and Elena are two biologists studying the population of frogs and butterflies that live in an island. Upon their arrival to the island, they found that there were 2 thousand frogs in the island. Show all your work.

a. [3 points] Luis believed that the population of frogs living in the island increases by 300 frogs every six months. Let $f(t)$ be the amount of frogs (in thousands) living in the island, t months after they arrived at the island, according to Luis belief. Find a formula for $f(t)$.

Solution: Since $f(t)$ is linear, then $f(t) = mt + b$. We know that the slope m of $f(t)$ is $\frac{0.3}{6}$ (0.3 thousand every month). Since there were 2 thousand frogs in the island when they arrived, then $b = 2$. Hence $f(t) = 2 + 0.05t$.

b. [3 points] Elena's hypothesis is that the population of frogs living in the island increases exponentially at a rate of 23% every month. Let $g(t)$ be the amount of frogs (in thousands) living in the island, t months after they arrived at the island, according to Elena's hypothesis. Find a formula for $g(t)$.

Solution: The function $g(t)$ is exponential, then $g(t) = ab^t$. Since there were 2 thousand frogs in the island when they arrived, then $a = 2$. Since the population of frogs living in the island increases exponentially at a rate of 23% every month, then $b = 1+0.23 = 1.23$. Hence $f(t) = 2(1.23)^t$

As the frog's population increased, the amount of butterflies in the island started to decrease. The population of butterflies 2 and 5 months after Elena and Luis arrived at the island was 20 thousand and 7 thousand respectively.

c. [4 points] Let $G(t)$ be a linear function describing the population of butterflies (in thousands) t months after the biologists arrive at the island. Find a formula for $G(t)$.

Solution: The problem states that the points $(2, 20)$ and $(5, 7)$ are in the graph of the linear function $G(t) = mt + b$. The slope of G is $m = \dfrac{7-20}{5-2} = -\dfrac{13}{3}$.
Using the point $(2, 20)$, we find

$$G(t) = 20 - \frac{13}{3}(t-2) = -\frac{13}{3}t + \frac{86}{3}.$$

The problem continues on the next page

The statement of the problem has been included for your convenience

As the frog's population increased, the amount of butterflies in the island started to decrease. The population of butterflies 2 and 5 months after Elena and Luis arrived at the island was 20 thousand and 7 thousand respectively.

d. [5 points] Let $H(t)$ be an exponential function describing the population of butterflies (in thousands) t months after the biologists arrive at the island. Find a formula for $H(t)$. Your answer must be in **exact form**.

Solution: The problem states that the points $(2, 20)$ and $(5, 7)$ are in the graph of the exponential function $H(t) = ab^t$. Hence a and b satisfy $20 = ab^2$ and $7 = ab^5$. Then

$$\frac{ab^5}{ab^2} = \frac{7}{20}$$
$$b^3 = \frac{7}{20} = 0.35$$
$$b = \sqrt[3]{0.35}.ab^2 = 20$$
$$a = \frac{20}{(\sqrt[3]{0.35})^2} \qquad f(t) = ab^t = \frac{20}{(\sqrt[3]{0.35})^2}\left(\sqrt[3]{0.35}\right)^t$$

e. [2 points] By what percentage is the population of butterflies reduced every month? Your answer must be accurate up to the first two decimals.

Solution: It is reduced by 29.53% every month.

7. [11 points] In a small isolated island, the local government has decided to start a recycling program. Consider the following functions:

- Let $F(r)$ be the amount of money (in millions of dollars) that the local government has to spend in order to recycle r tons of garbage.
- Let $G(p)$ be the amount of recyclable garbage (in tons) the island generates in a year when there are p thousands of people living in the island.
- Let $H(t)$ be the amount of people (in thousands) living in the island t years after 2010.

Assume that the functions F, G and H have inverses.

a. [6 points] Find a practical interpretation to the following mathematical expressions:

i) $F(3) = 2$

Solution: The government spends 2 million dollars recycling 3 tons of garbage.

ii) $G(H(4))$

Solution: The amount of recyclable garbage (in tons) the island generates in 2014.

b. [1 point] Let A be the average rate of change of the function G for $3 \leq p \leq 5$. What are the units of A?

Solution: Units of A= tons per thousand of people.

c. [4 points] Fill in the blanks in the following statements using the correct mathematical expression. A list of possible answers are listed below. Write your own expression if the correct expression is not on the list.

Solution:

i) The government spends 25 millions of dollars to recycle $\mathbf{F^{-1}(25)}$ tons of garbage.

ii) There were $\mathbf{G^{-1}(F^{-1}(25))}$ thousand people living in the island when the local government spent 25 million dollars recycling garbage.

8. [10 points] A cannon fires a cannonball. Let p be a positive constant and

$$f(t) = -5t^2 + pt + 30$$

be the height of the cannonball (in meters) above the ground t seconds after the cannon was fired.

a. [3 points] Find the value and a practical interpretation of the vertical intercept of the function $f(t)$.

Solution: Vertical intercept: $f(0) = 30$ meters.

Practical interpretation: The height of the cannon above the ground in meters.

b. [5 points] Complete the square to put the formula of f in vertex form. *Carefully show your algebraic work step by step.* Your answer may include the constant p.

Solution:

$$\begin{aligned}
f(t) &= -5t^2 + pt + 30 \\
&= -5\left(t^2 - \frac{p}{5}t\right) + 30 \\
&= -5\left[t^2 - \frac{p}{5}t + \left(\frac{p}{10}\right)^2 - \left(\frac{p}{10}\right)^2\right] + 30 \\
&= -5\left[\left(t - \frac{p}{10}\right)^2 - \left(\frac{p}{10}\right)^2\right] + 30 \\
&= -5\left(t - \frac{p}{10}\right)^2 + 5\left(\frac{p}{10}\right)^2 + 30 \\
&= -5\left(t - \frac{p}{10}\right)^2 + \frac{p^2}{20} + 30 = -5\left(t - \frac{p}{10}\right)^2 + \frac{p^2 + 600}{20}
\end{aligned}$$

c. [2 points] What should be the value of p if the maximum height of the cannonball is 200 meters above the ground? Find your answer algebraically. Show all your work.

Solution: The maximum height is given by $H_{max} = \dfrac{p^2 + 600}{20}$. Hence the maximum height of the cannonball will be 200 meters if $\dfrac{p^2 + 600}{20} = 200$. Hence $p^2 + 600 = 4000$ and $p = \sqrt{3400}$.

9. [12 points] A store sells socks. Let $S(p)$ be the profit (in dollars) the store earns from selling socks at a price of p dollars.

a. [5 points] The store manager notices that if they sell socks at 4 dollars, they get the highest profit of 2,500 dollars. If they sell socks at 2.50 dollars the profit is 1375 dollars. Suppose $S(p)$ is a quadratic function. Find a formula for $S(p)$.

Solution: The highest point on the graph of the profit function $S(p)$ is at the point $(4, 2500)$. Hence this is the vertex of the quadratic function $S(p)$.Then the vertex form of the function is $S(p) = a(p - 4)^2 + 2500$. Since $S(2.5) = 1375$, then $1375 = a(2.5 - 4)^2 + 2500$. This yields $a = -\frac{1125}{(1.5)^2} = -500$. Then $S(p) = -500(p - 4)^2 + 2500$.

b. [7 points] The winter season is here and the store is now selling mittens. Let M be the profit (in dollars) the store earns from selling mittens at a price of p dollars, where

$$M = f(p) = -96(p - 3.5)^2 + 600.$$

i) At what price(s) will the store not have any profit from selling mittens? You must find your answer algebraically.

Solution: We need to find the prices p at which $f(p) = -96(p - 3.5)^2 + 600 = 0$.

$$\begin{aligned} -96(p-3.5)^2 + 600 &= 0 \\ (p-3.5)^2 &= \frac{600}{96} = 6.25 \\ p - 3.5 &= \pm\sqrt{6.25} = \pm 2.5 \\ p = 6 \qquad & p = 1. \end{aligned}$$

ii) Suppose that the store refuses to sell any mittens at any price at which no profit will be obtained. What is the practical domain and range of the function $f(p)$? Use interval notation or inequalities to answer this question. Your answer needs to be in *exact form* or be accurate up to two decimals.

Solution: The vertex $(3.5, 600)$ represents the maximum point in the graph of $f(p)$. Since the prices will never yield non positive profits, then the practical range of $f(p)$ is $(0, 600]$. The domain is the prices that yield a positive profit. Then the domain of $f(p)$ is $1 < p < 6$.