Math 105 — Second Midterm

March 19, 2015

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	16	
2	11	
3	6	
4	10	
5	15	
6	9	
7	11	
8	13	
9	9	
Total	100	

1. [16 points]

a. [10 points] Indicate if each of the following statements are true or false by circling the correct answer. No justification is required.

i) The graph of $f(x) = 10^{x+1}$ can be obtained by applying a vertical stretch by 10 to the graph of the function $g(x) = 10^x$.

[True] False

ii) An angle of 180 radians corresponds to a half rotation around the unit circle.

True [False]

iii) If $y = (\ln(x))^{\frac{1}{2}}$ then $y = \frac{1}{2}\ln(x)$.

True [False]

iv) The function $f(x) = 5^{3x} + 5^{-3x}$ is an odd function.

True [False]

v) For any $x > 0$, $e^{2\ln(x)} = x^2$.

[True] False

b. [6 points]

i) Find the equation of the vertical asymptote of the function $f(x) = 5\log(7x + 2)$. Your answer must be exact.

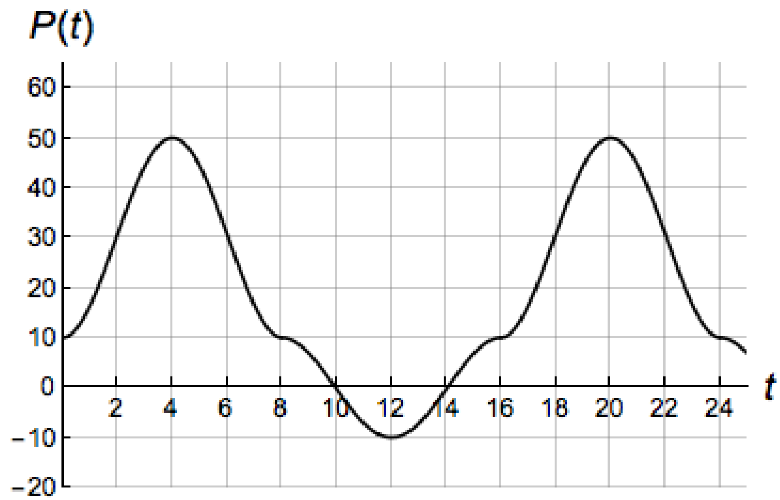
Solution: Vertical asymptote of $f(x)$ at $\mathbf{x} = -\mathbf{\frac{2}{7}}$

ii) Let $g(x) = 1 - \log(x + 2)$. Compute the following quantities:

$\lim_{x\to-2^+} g(x) = \infty$

$\lim_{x\to\infty} g(x) = -\infty$

2. [11 points] Let $P(t)$ be the average temperature (in ^{0}F) in a small moon that rotates around a planet at time t (in hours). Suppose that $P(t)$ is a periodic function with period less than 20 hours. The graph of $y = P(t)$ is shown below



a. [2 points] Find the period of $P(t)$:

Solution: Period of $P(t)$ =16.

b. [2 points] Find the amplitude of the function $P(t)$:

Solution: Amplitude=$\frac{50-(-10)}{2} = 30^o$ F

c. [2 points] Find the equation of the midline of the function $P(t)$:

Solution: Midline: $y = \frac{50+(-10)}{2} = 20$.

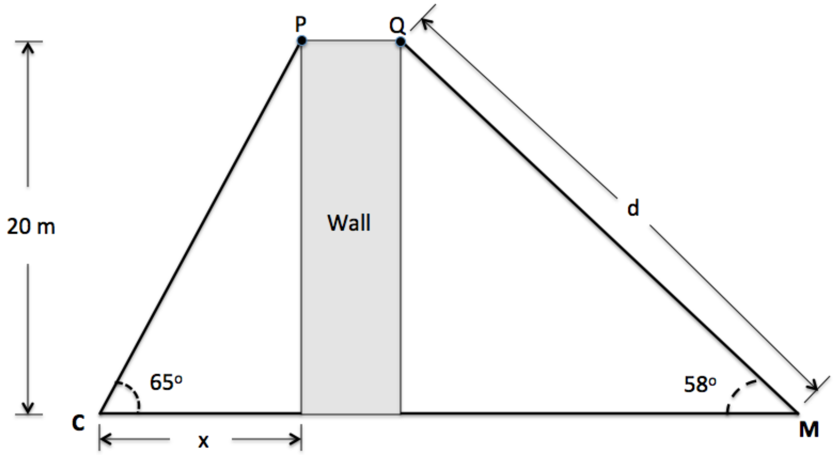
d. [3 points] What is the smallest value of t that satisfies $t > 24$ and $P(t) = 30$?

Solution: The solutions to $P(t) = 30$ for $0 < t < 24$ are (from the graph) $t = 2, 6, 18$ and 22. Hence the next solution is at $t = 18 + 16 = 34$ hours.

e. [2 points] Let $k(t) = 2P(3t)$. What is the period of the function $k(t)$?

Solution: The graph of $k(t)$ can be obtained from the graph of $P(t)$ by applying a vertical stretch by 2 and a horizontal compression by $\frac{1}{3}$. The only transformation that determines the period of $k(t)$ is the horizontal compression, then the period of $k(t)$ is $\frac{16}{3}$ hours.

3. [6 points] Casey and Milan are standing at different sides of a 20 meter high wall, at the points C and M respectively. They measured the angles determined by the points P,C,M and Q,M, C and found that they were 65 and 58 degrees respectively (see the figure below).



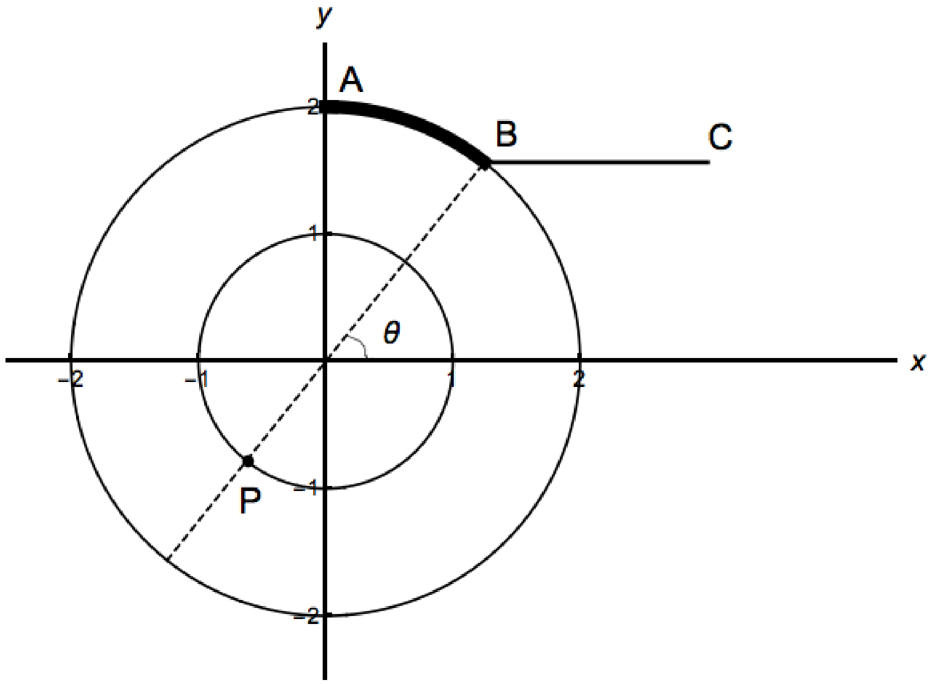
i) Find the distance d between the points M and Q. Your answer needs to be exact or rounded up to the nearest .01. Show all your work.

Solution: Using $\sin(58^o) = \dfrac{20}{d}$, we get $d = \dfrac{20}{\sin(58^o)}$ meters.

ii) Find the distance x between Casey's position and the wall. Your answer needs to be exact or rounded up to the nearest .01. Show all your work.

Solution: Using $\tan(65^o) = \dfrac{20}{x}$, we get $x = \dfrac{20}{\tan(65^o)}$ meters.

4. [10 points] In the picture below, there are two circles of radius 1 and 2 centered at the origin. The point A has coordinates $(0, 2)$. The point B is the intersection of the line passing throughout the point $(0, 0)$ and P. The angle θ formed by the line joining $(0, 0)$ and the point B with the positive side of the x-axis is measured in **radians**.



Find a formula for the following quantities. Your answers may depend on the angle θ, but not on any of the constants a, b, c or d.

a) Let $B = (a, b)$.

Solution: Find $b = 2\sin(\theta)$.

b) Let $P = (c, d)$.

Solution: Find $c = \cos(\theta + \pi) = -\cos(\theta)$.

c) Find the length of the arc defined by the points A and B (the arc is shown in bold in the figure above).

Solution: Length of the arc $= 2(\frac{\pi}{2} - \theta) = \pi - 2\theta$.

d) The point C has coordinates $(3, b)$. Find the length of the line segment BC.

Solution: Length of BC $= 3 - 2\cos(\theta)$.

5. [15 points]

a. [6 points] Consider the function

$$F(x) = \begin{cases} x^2 & \text{for} \quad 0 \le x \le 2 \\ 7 + \dfrac{x}{5} & \text{for} \quad 2 < x \le 4. \end{cases}$$

Find a piecewise defined formula for the function $G(x) = 3F(x-1)$.

Solution:

$$G(x) = \begin{cases} 3(x-1)^2 & \text{for} \quad 1 \le x \le 3 \\ 3\left(7 + \dfrac{x-1}{5}\right) & \text{for} \quad 3 < x \le 5. \end{cases}$$

b. [5 points] Some of the values of the functions $H(x)$, $K(x)$ and $J(x)$ are shown in the tables below

x	0	1	2	3
$H(x)$	1	-2	3	-4

x	-2	-1	0	1
$J(x)$	2	-1	4	-3

x	0	2	4	6
$K(x)$	-1	2	-3	4

i) The functions $J(x)$ and $K(x)$ are obtained by applying transformations to the function $H(x)$. Find a possible formula for $J(x)$ and $K(x)$. A list of possible answers is shown below. If the answer is not included in the list write your own formula for it in terms of transformations of the function $H(x)$.

Solution: $\mathbf{J(x) = H(x+2) + 1}$ $\mathbf{K(x) = -H\left(\frac{1}{2}x\right)}$

$2H(x-2)$ $H(x+2)+1$ $2H(x+2)$ $H(x+2)+1$

$H(-2x)$ $-H\left(\frac{1}{2}x\right)$ $-H(2x)$ $H\left(-\frac{1}{2}x\right)$

ii) Suppose $H(x)$ is an even function, what is the value of $H(-3)$?

Solution: $H(-3) = -4$

c. [4 points] The graph of a function $w = f(z)$ contains the point $(5, -1)$ and has a horizontal asymptote at $w = 2$. Let $g(z) = 1 - f\left(\frac{z}{2} + 3\right)$. Find a point in the graph $g(z)$ and the equation of its horizontal asymptote.

Solution: The graph of $g(z)$ contains the point $(\mathbf{4}, \mathbf{2})$.
Equation of the horizontal asymptote of $g(z)$ at $\mathbf{w = -1}$.

6. [9 points] Let $V(t)$ be the total number of tickets for a concert that have been sold (in thousands) t minutes after 8 pm.

a. [2 points] There are only 2 million tickets for the concert on sale. Let $U(t)$ be the number of unsold tickets at t minutes after 8 pm. Find a formula for $U(t)$.

Solution: $U(t) = 2,000,000 - 1000V(t)$

b. [2 points] Let $E(m)$ be the total number of tickets for the concert that have been sold (in thousands) m minutes **after 11 pm**. Find a formula for $E(m)$.

Solution: $E(m) = V(m + 180)$

c. [2 points] Let $H(p)$ be the total number of tickets for the concert that have been sold (**in hundreds**) p **hours** after 8 pm. Find a formula for $H(p)$.

Solution: $H(p) = 10V(60p)$

d. [3 points] Write an equation that represents the following fact: *"M minutes after 8 pm, the number of tickets sold was equal to a third of the tickets sold by 9 pm".*

Solution: Equation: $V(M) = \frac{1}{3}V(60)$.

7. [11 points] Ammonia was spilled into two lakes. Companies A and B are hired to clean the first and second lakes, respectively. Let $A(t)$ be the amount of ammonia (in gallons) in the first lake t hours after company A starts cleaning the lake, where

$$A(t) = 2300(0.78)^t.$$

a. [2 points] How many gallons of ammonia were spilled in the first lake?

Solution: Answer: 2300 gallons.

b. [2 points] What is the continuous rate per hour of the function $A(t)$? Your answer needs to be in exact form.

Solution: Using the fact that $e^k = 0.78$, we get $k = \ln(0.78)$.

c. [3 points] How long does it takes to reduce the amount of ammonia by 20 percent of its initial amount? Show all your work. Your answer needs to be in exact form. Include units.

Solution:

$$\begin{aligned} 2300(0.78)^t &= 0.8(2300) \\ (0.78)^t &= 0.8 \\ \ln((0.78)^t) &= \ln(0.8) \\ t\ln(0.78) &= \ln(0.8) \qquad t = \frac{\ln(0.8)}{\ln(0.78)} \text{ hours.} \end{aligned}$$

Company B states that their cleaning method is capable of reducing the amount of ammonia in the lake at a continuous rate of 30 percent every hour.

d. [3 points] Let $B(t)$ be the amount of ammonia (in gallons) in the second lake t hours after company B starts cleaning the second lake. Find a formula for $B(t)$ if the amount of ammonia spilled in the second lake is 3 thousand gallons.

Solution: $B(t) = 3000e^{-0.3t}$

e. [1 point] Which company reduces the amount of ammonia faster? Circle your answer.

Solution: Since the continuous rate per hour of company A, $k = \ln(0.78) \approx -0.248$ is larger than the continuous rate per hour of company B, $k = -0.3$. Then the method from company B reduces the ammonia faster.

8. [13 points] Solve the following equations algebraically. Your solutions should be in **exact form**. Show all your work to receive full credit.

a. [4 points] $\log(2x) - \log(2x+1) = 3$.

Solution:

$$\begin{aligned}
\log(2x) - \log(2x+1) &= 3 \\
\log\left(\frac{2x}{2x+1}\right) &= 3 \\
\frac{2x}{2x+1} &= 10^3 \\
2x &= 1000(2x+1) \\
2x &= 2000x + 1000 \\
-1998x &= 1000 \\
x &= -\frac{1000}{1998} \qquad \text{No solution.}
\end{aligned}$$

b. [4 points] $\ln(6e^{2x} - 8) = 2x$

Solution:

$$\begin{aligned}
\ln(6e^{2x} - 8) &= 2x \\
6e^{2x} - 8 &= e^{2x} \\
5e^{2x} &= 8 \\
e^{2x} &= \frac{8}{5} \\
2x &= \ln\left(\frac{8}{5}\right) \qquad x = \frac{1}{2}\ln\left(\frac{8}{5}\right)
\end{aligned}$$

c. [5 points] $1.3(10^x) = (3.4)^{3x}$

Solution:

$$\begin{aligned}
1.3(10^x) &= (3.4)^{3x} \\
\log(1.3(10^x)) &= \log((3.4)^{3x}) \\
\log(1.3) + \log(10^x) &= 3x\log(3.4) \\
\log(1.3) + x &= 3x\log(3.4) \\
\log(1.3) &= 3x\log(3.4) - x \\
\log(1.3) &= x(3\log(3.4) - 1) \\
x &= \frac{\log(1.3)}{3\log(3.4) - 1}.
\end{aligned}$$

9. [9 points] A new computer antivirus is available to be downloaded on March 19 at the university's ITS website. The antivirus is only available for students. Let $T(s)$ be the time (in days after March 19) it takes for s students to download the antivirus to their personal computers. The function $T(s)$ is given by

$$T(s) = 20\log\left(\frac{s}{2}+1\right).$$

a. [3 points] List the transformations required to obtain the graph of $T(s)$ from the graph of the function $f(s) = \log(s)$. Make sure to be precise when you describe each transformation and indicate the order in which they need to be applied.

Solution:

1. Vertical Stretch by 20.
2. Horizontal Shift to the left by 1.
3. Horizontal Stretch by 2.

b. [2 points] How many days after March 19 are required for a thousand students to download the antivirus to their personal computers? Your answer needs to be exact or rounded up to the nearest .01.

Solution:

$$T(1000) = 20\log\left(\frac{1000}{2}+1\right) = 20\log(501) \text{ days after March 19.}$$

c. [4 points] How many students have downloaded the antivirus seven days after March 19? Your answer must be found algebraically and written in exact form.

Solution:

$$\begin{aligned} 20\log\left(\frac{s}{2}+1\right) &= 7. \\ \log\left(\frac{s}{2}+1\right) &= \frac{7}{20} \\ \frac{s}{2}+1 &= 10^{\frac{7}{20}} \\ \frac{s}{2} &= 10^{\frac{7}{20}} - 1 \\ s &= 2(10^{\frac{7}{20}} - 1). \end{aligned}$$