Math 105 — Final Exam

April 23, 2015

Name: EXAM SOLUTIONS

Instructor: ____________________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	10	
3	11	
4	14	
5	15	
6	13	
7	5	
8	9	
9	12	
Total	100	

1. [11 points]

a. [8 points] Indicate if each of the following statements are true or false by circling the correct answer. No justification is required.

i) If $f(3) = 4$ then the point $(4, 3)$ is on the graph of $y = f^{-1}(x)$.

[True] False

ii) If a polynomial $p(x)$ has odd degree, then the function $p(x)$ is an odd function.

True **[False]**

iii) If the function $f(x)$ is odd, then the function $g(x) = xf(x)$ is an even function.

[True] False

iv) The function $h(x) = 2 - (x - 4)^2$ with domain $x \geq 4$ is an invertible function.

[True] False

b. [2 points] Compute the value of the following limits:

Solution: $\lim_{x \to -\infty} \frac{2e^x + 1}{5 + x} = 0$ $\lim_{x \to -3^-} \frac{-1}{x + 3} = \infty$

c. [1 point] Let $f(x) = x^{\frac{1}{5}}$ and $g(x) = 1 + \log(x)$. Which of the functions grows more rapidly as $x \longrightarrow \infty$? Circle your answer.

Solution:

[$f(x)$] $g(x)$ It can't be determined.

2. [10 points]

a. [5 points] Let $f(x)$ be a function that satisfies all of the following statements:

a) The domain of $f(x)$ is $[-4, 5)$.

b) The graph of $y = f(x)$ has only one horizontal intercept at $x = 2$.

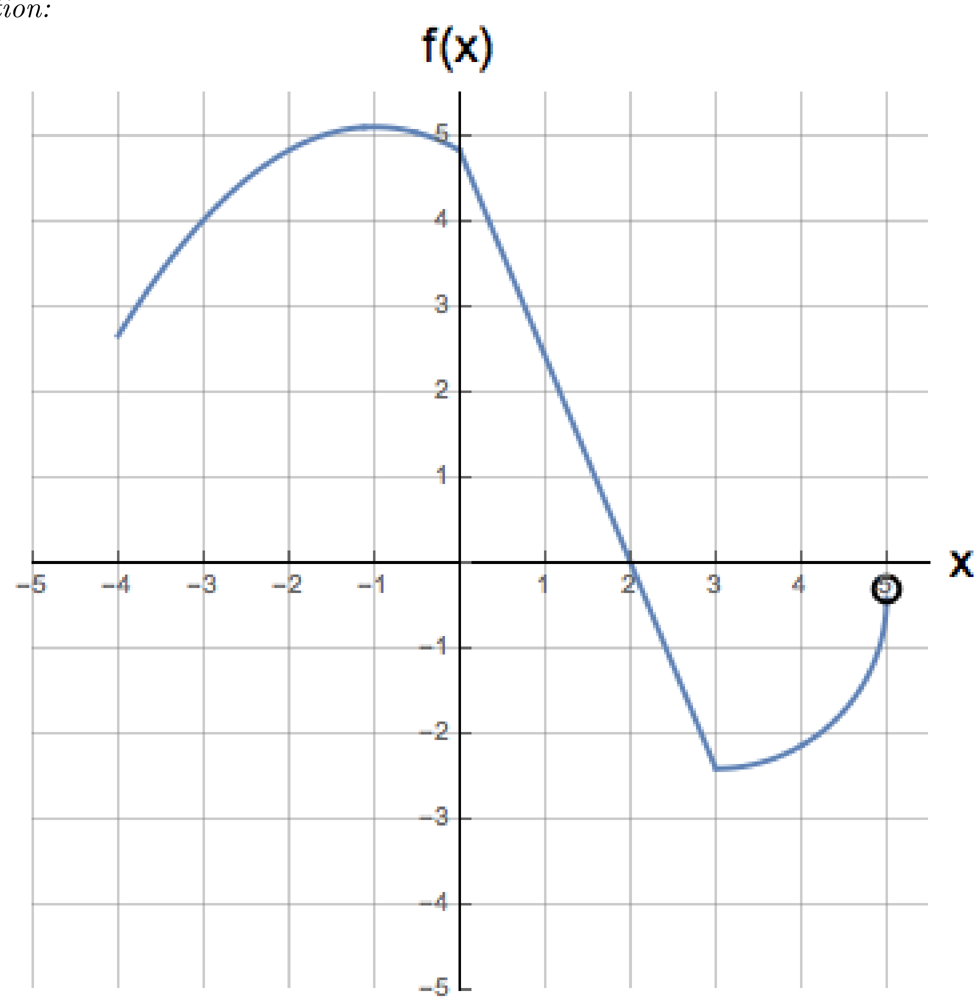
c) The function $f(x)$ is decreasing for $-1 \leq x \leq 3$.

d) The function $f(x)$ is concave down for $-4 \leq x \leq 0$ and concave up for $3 \leq x < 5$. Make sure the concavity of $f(x)$ is clear in your graph.

e) The function $f(x)$ has constant rate of change for $0 \leq x \leq 3$.

Draw a possible graph for $f(x)$. Make sure to label the important points on the graph to receive full credit.

Solution:



b. [5 points] Let $w = K(r)$, where $K(r) = \log\left(7e^{2r} + 4\right) + 5$. Find a formula for $K^{-1}(w)$. Show all your work.

Solution:

$$\begin{aligned} w &= \log\left(7e^{2r} + 4\right) + 5 \\ w - 5 &= \log\left(7e^{2r} + 4\right) \\ 10^{w-5} &= 7e^{2r} + 4 \\ 7e^{2r} &= 10^{w-5} - 4 \\ e^{2r} &= \frac{10^{w-5} - 4}{7} \\ 2r &= \ln\left(\frac{10^{w-5} - 4}{7}\right) \\ K^{-1}(w) &= \frac{1}{2}\ln\left(\frac{10^{w-5} - 4}{7}\right) \end{aligned}$$

3. [11 points]

a. [4 points] A store sells bananas. For an order of less than 30 pounds of bananas, the store charges 45 cents per pound of bananas. On a purchase of 30 pounds or more, the store charges 34 cents per pound of bananas plus an additional fee of 3.30 dollars for packaging costs. A minimum of one pound of bananas is required on every purchase. Let $C(x)$ be the cost (in **dollars**) of buying x pounds of bananas. Find a formula for $C(x)$ as a piecewise-defined function.

Solution:

$$C(x) = \begin{cases} 0.45x & \text{for} \quad 1 \le x < 30 \\ \\ 3.30 + 0.34x & \text{for} \quad 30 \le x \end{cases}$$

b. [7 points] There is a fire in the forest. The amount of forest burnt (in km^2) increases exponentially at a continuous growth rate of 40 percent per hour. Authorities estimate that 12 km^2 of forest were burnt half an hour after the fire started. Let $B(t)$ be the total area burnt by the fire (in km^2) t hours after the fire started.

i) Find a formula formula for $B(t)$. Your formula has to be written in **exact form**.

Solution: The function $B(t) = ae^{0.4t}$ satisfies $B(0.5) = 12$, then

$$12 = ae^{0.2}$$
$$a = \frac{12}{e^{0.2}}$$

Then $B(t) = \dfrac{12}{e^{0.2}}e^{0.4t} = 12e^{0.4t-0.2}$.

ii) What is the hourly percent rate at which the fire burns the forest? Your answer must be written in **exact form** or accurate up to the first three decimals.

Solution: Since $b = e^{0.4} = 1 + r$, then $r = e^{0.4} - 1 \approx .492$ (or $100(e^{0.4} - 1) \approx 49.182$ percent).

iii) What is the doubling time of the function $B(t)$? Your answer must be found algebraically and written in **exact form**. Show all your work.

Solution:

$$2a = ae^{0.4t}$$
$$e^{0.4t} = 2$$
$$t = \frac{1}{0.4}\ln 2 \text{ hours}$$

4. [14 points] The noise level of the sound of a fire alarm at a factory oscillates between a maximum of 120 decibels to a minimum of 70 decibels. It takes the alarm 10 seconds to go from its maximum to its minimum noise level. Let $y = f(t)$ be the noise level of the sound of the alarm (in decibels) t seconds after it is activated. Suppose that $f(t)$ is a **sinusoidal** function and that $f(0) = 70$.

a. [6 points] Find the period, the amplitude, the midline and formula of the function $y = f(t)$. Include units.

Solution: Period= 20 seconds Amplitude= $\frac{120-70}{2} = 25$ decibels

Midline: $y = \frac{120+70}{2} = 95$

$f(t) = 95 - 25\cos\left(\frac{\pi}{10}t\right)$

b. [2 points] The factory considered the alarm to be too loud. They modified their alarm by reducing its noise level by 20 percent and they doubled the length of time it takes for the alarm to go from its lowest to its maximum noise level. Let $g(t)$ be the noise level (in decibels) of the modified alarm t seconds after it is activated. Find a formula for $g(t)$ in terms of the function $f(t)$ or as a sinusoidal function in terms of t.

Solution: $g(t) = 0.8f\left(\frac{1}{2}t\right)$.

c. [6 points] The noise level $F(t)$ of the sound (in decibels) of a fire alarm at a different factory t seconds after it is activated is given by

$$F(t) = 80 + 35\sin\left(\frac{\pi}{28}t\right)$$

It is known that sounds with intensity above 110 decibels are dangerous for the human ear. For which values of $0 \le t \le 70$ will the intensity of the fire alarm be exactly 110 decibels? *You must find your answer(s) algebraically and write them in **exact form**. Show all your work.*

Solution:

$$110 = 80 + 35\sin\left(\frac{\pi}{28}t\right)$$
$$\sin\left(\frac{\pi}{28}t\right) = \frac{30}{35}$$
$$t = \frac{28}{\pi}\sin^{-1}\left(\frac{30}{35}\right)$$
$$t_1 = \frac{28}{\pi}\sin^{-1}\left(\frac{30}{35}\right) \qquad t_2 = 28 - t_1 \qquad t_3 = t_1 + 56.$$

5. [15 points] Consider the functions $f(x)$, $g(x)$ and $j(x)$ given by the tables below

x	-1	1	3	5
$f(x)$	11.1	6.5	4.1	3
$g(x)$	2.4	1.7	1	0.3
$j(x)$	0.25	0.5	1	2

Assume that all the functions above are invertible.

a. [2 points] Which function(s) could be concave up? Circle all possible answers.

Solution: $\boxed{f(x)}$ $g(x)$ $\boxed{j(x)}$ None of these

b. [2 points] Which function(s) could be a linear function? Circle all possible answers.

Solution: $f(x)$ $\boxed{g(x)}$ $j(x)$ None of these

c. [2 points] Which function(s) could be an exponential function? Circle all possible answers.

Solution: $f(x)$ $g(x)$ $\boxed{j(x)}$ None of these

d. [4 points] Compute the value of the following quantities. If there is not enough information to compute the values write "Undefined".

Solution: $g(f^{-1}(3)) = g(5) = 0.3$ $\quad (j(g(3)))^{-1} = (j(1))^{-1} = (0.5)^{-1} = 2$

e. [3 points] Let $Q(t) = 3t^2 + 1$ and h be a constant. Find a simplified formula for $\dfrac{Q(t+h) - Q(t)}{h}$. Your answer may depend on t and h.

Solution:

$$\frac{Q(t+h) - Q(t)}{h} = \frac{3(t+h)^2 + 1 - (3t^2+1)}{h} = \frac{6th + 3h^2}{h} = 6t + 3h$$

f. [2 points] Let $H(x) = \cos(1 + 2\log(x))$ and $G(x) = \log(x)$. Find a function $F(x)$ such that $H(x) = F(G(x))$.

Solution: $F(x) = \cos(1 + 2x)$

6. [13 points]

a. [6 points] Let $P(x) = \dfrac{(x^2 - 2x - 4)^2}{7x^4 - 1400}$.

Find the following quantities. Your answers must be found algebraically and written in **exact form**. Show your work.

i) Find the equation(s) of the vertical asymptotes of $P(x)$:

Solution: Vertical asymptotes when $7x^4 - 1400 = 0$ or $x = \pm\sqrt[4]{200}$.

ii) Find the horizontal intercepts of $P(x)$.

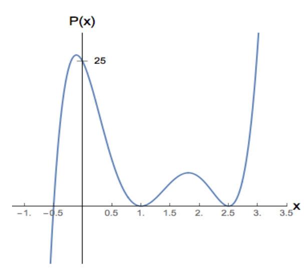
Solution: Horizontal intercepts when $x^2 - 2x - 4 = 0$. Using the quadratic formula

$$x = \frac{2 \pm \sqrt{4 - 4(-4)}}{2} = 1 \pm \sqrt{5}.$$

iii) Does the graph of $y = P(x)$ have horizontal asymptotes? If so, write its equation, otherwise write "None".

Solution: Since the function $P(x)$ behaves as $x \to \pm\infty$ similarly to $\frac{x^2)^2}{7x^4} = \frac{1}{7}$, then the graph of $P(x)$ has a horizontal asymptote at $y = \dfrac{1}{7}$.

b. [5 points] Find the formula for the polynomial $P(x)$ of degree five shown below



Solution: Looking at the graph $P(x) = a(x + 0.5)(x - 1)^2(x - 2.5)^2$. Since $P(0) = 25$, then $25 = a(0.5)(-1)^2(-2.5)^2$. This yields $a = 8$.

Hence $P(x) = 8(x + 0.5)(x - 1)^2(x - 2.5)^2$

c. [2 points] Write the equation of a rational function $R(x)$ that satisfies both conditions below:

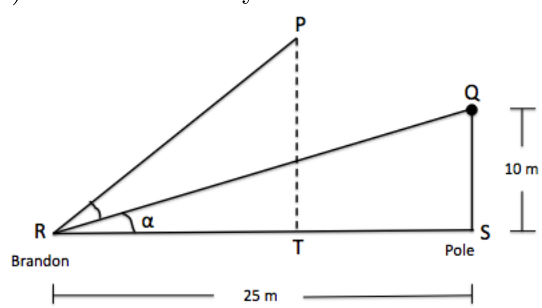
i) The graph of $y = R(x)$ has a vertical asymptote at $x = 1$.

ii) $\lim_{x\to\infty} R(x) = \infty$

There may be more than one correct answer. You only need to find one of them.

Solution: The rational function $R(x) = \dfrac{p(x)}{q(x)}$ has to be undefined at $x = 1$ and the degree of $P(x)$ has to be larger than the degree of $q(x)$. One example of a rational function satisfying both conditions is $R(x) = \dfrac{x^2}{x-1}$.

7. [5 points] Brandon takes a picture of a bird standing (at point Q) on top of a 10 meter high pole. The pole (at point S) is 25 meters away from where Brandon stands (at point R).



i) Find the value of the angle α (the angle SRQ) measured in **radians**. *Your answer must be written in* ***exact form*** *or accurate up to two decimals. Show all your work.*

Solution: Since $\tan(\alpha) = \frac{10}{25}$ then $\alpha = \tan^{-1}\left(\dfrac{10}{25}\right) \approx 0.3805$.

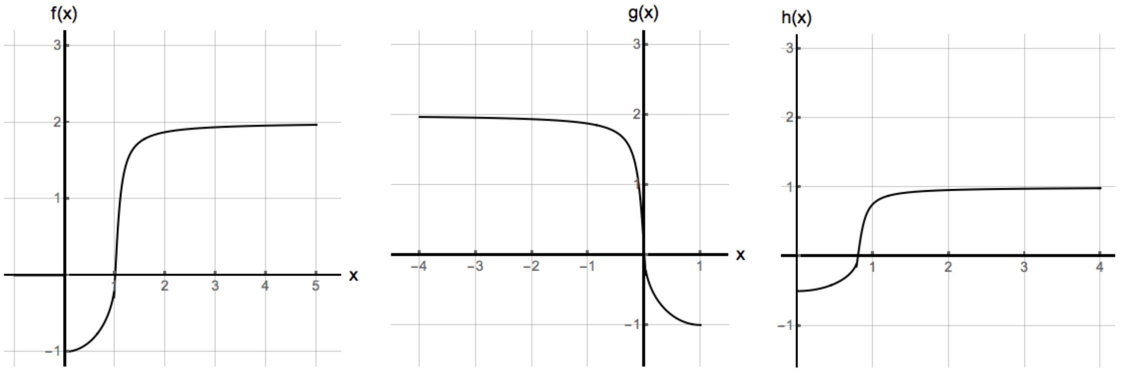
ii) The bird flies to the top of a nearby tree (point P). Brandon takes a second picture of the bird once it is at the top of the tree. The length of the segment RP is 20 meters. What is the height of the tree, the length of the segment PT, if you know that the angle QRP measures 0.75 radians.

Your answer must be written in ***exact form*** *or accurate up to two decimals. Show all your work.*

Solution: In this case $\sin(\alpha + 0.75) = \dfrac{PT}{20}$, then $PT = 20\sin(\alpha + 0.75) \approx 18.092$

8. [9 points]

a. [4 points] Let $f(x)$, $g(x)$ and $h(x)$ be the functions shown below



Find formulas for the functions $g(x)$ and $h(x)$ as transformations of the function $f(x)$. A list of possible answers is shown below. If the correct answer is not included in the list, write your own formula in terms of transformations of the function $f(x)$.

$\frac{1}{2}f\left(\frac{4}{5}x\right)$	$-f(x+1)$	$f(-x-1)$	$2f\left(\frac{5}{4}x\right)$
$f(-x+1)$	$2f\left(\frac{4}{5}x\right)$	$\frac{1}{2}f\left(\frac{5}{4}x\right)$	$-f(x-1)$

Solution: $g(x) = f(-x+1)$ $\qquad h(x) = \dfrac{1}{2}f\left(\dfrac{5}{4}x\right)$

b. [5 points] A bookstore keeps an e-mail list of its regular customers. The list had 250 and 750 e-mail addresses in 2004 and 2010 respectively. Let $M(t)$ be the number of e-mail addresses in the list t years after 2000. Suppose $M(t)$ is a power function. Find a formula for $M(t)$. Your answer must be written in **exact form**. Show all your work.

Solution: Let $M(t) = kt^p$, then the points $(4, 250)$ and $(10, 750)$ are in the graph of $M(t)$ then

$$250 = k(4)^p \qquad 750 = k(10)^p$$

$$3 = (2.5)^p \quad \text{then} \quad p = \frac{\ln(3)}{\ln(2.5)}$$

$$k = \frac{250}{4^p} = \frac{250}{4^{\frac{\ln(3)}{\ln(2.5)}}}$$

$$M(t) = \left(\frac{250}{4^{\frac{\ln(3)}{\ln(2.5)}}}\right) t^{\frac{\ln(3)}{\ln(2.5)}}$$

9. [12 points]

a. [6 points] Let $f(y)$ be the length of a trout (in inches) that is y years old and $g(d)$ be the weight (in lbs) of a trout of length d inches. Suppose that both f and g are invertible functions. Find a practical interpretation for the following mathematical expressions:

i) $g(17) = 3$

Solution: A trout that measures 17 inches weighs 3 lbs.

ii) $f^{-1}(7)$

Solution: The age of a trout in years that measures 7 inches.

iii) $g(f(7))$

Solution: The weight of a trout in lbs that is 7 in years old.

b. [6 points] Let $A(t)$ and $B(t)$ be the number of apple and pear trees in Michigan t years after 2005. Let $C(t)$ be the average harvest yield of apples per tree (in pounds per tree) in Michigan t years after 2005. Similarly, define $D(t)$ to be the average harvest yield of pears per tree (in pounds per tree) in Michigan t years after 2005. Find mathematical expressions using the functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$ for each of the following quantities:

i) The number of apple and pear trees in Michigan in 2013.

Solution: $A(8) + B(8)$

ii) The total number of pounds of apple harvested in Michigan in 2005.

Solution: $A(0)C(0)$

iii) The average harvest yield of pears per tree (in pounds per tree) in Michigan k **decades after 2010** (1 decade = 10 years).

Solution: $D(5 + 10k)$