Math 105 — First Midterm

February 9, 2016

Name: EXAM SOLUTIONS

Instructor: ______________________ Section: __________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

Problem	Points	Score
1	11	
2	8	
3	11	
4	13	
5	14	
6	11	
7	7	
8	14	
9	11	
Total	100	

1. [11 points] Let

x	-1	0	1	2	3
$f(x)$	20	35	60	85	100
$g(x)$	192	48	12	3	0.75
$h(x)$	-31	-20	-9	2	13

$$k(x) = 0.3(x-5)^2 - 3$$

a. [9 points] Answer the following questions using the functions given above.

Solution:

i) Which of the functions could be (or are) linear? Circle all that apply.

$f(x)$ $g(x)$ $\boxed{h(x)}$ $k(x)$ None.

ii) Which of the functions could be (or are) concave up? Circle all that apply.

$f(x)$ $\boxed{g(x)}$ $h(x)$ $\boxed{k(x)}$ None.

iii) Which of the functions could be (or are) exponential? Circle all that apply.

$f(x)$ $\boxed{g(x)}$ $h(x)$ $k(x)$ None.

iv) Which of the functions could be (or are) increasing? Circle all that apply.

$\boxed{f(x)}$ $g(x)$ $\boxed{h(x)}$ $k(x)$ None.

b. [2 points]

p	1	2	3	4	5
W	-2	4	1	3	1
Z	1	3	0	2	4

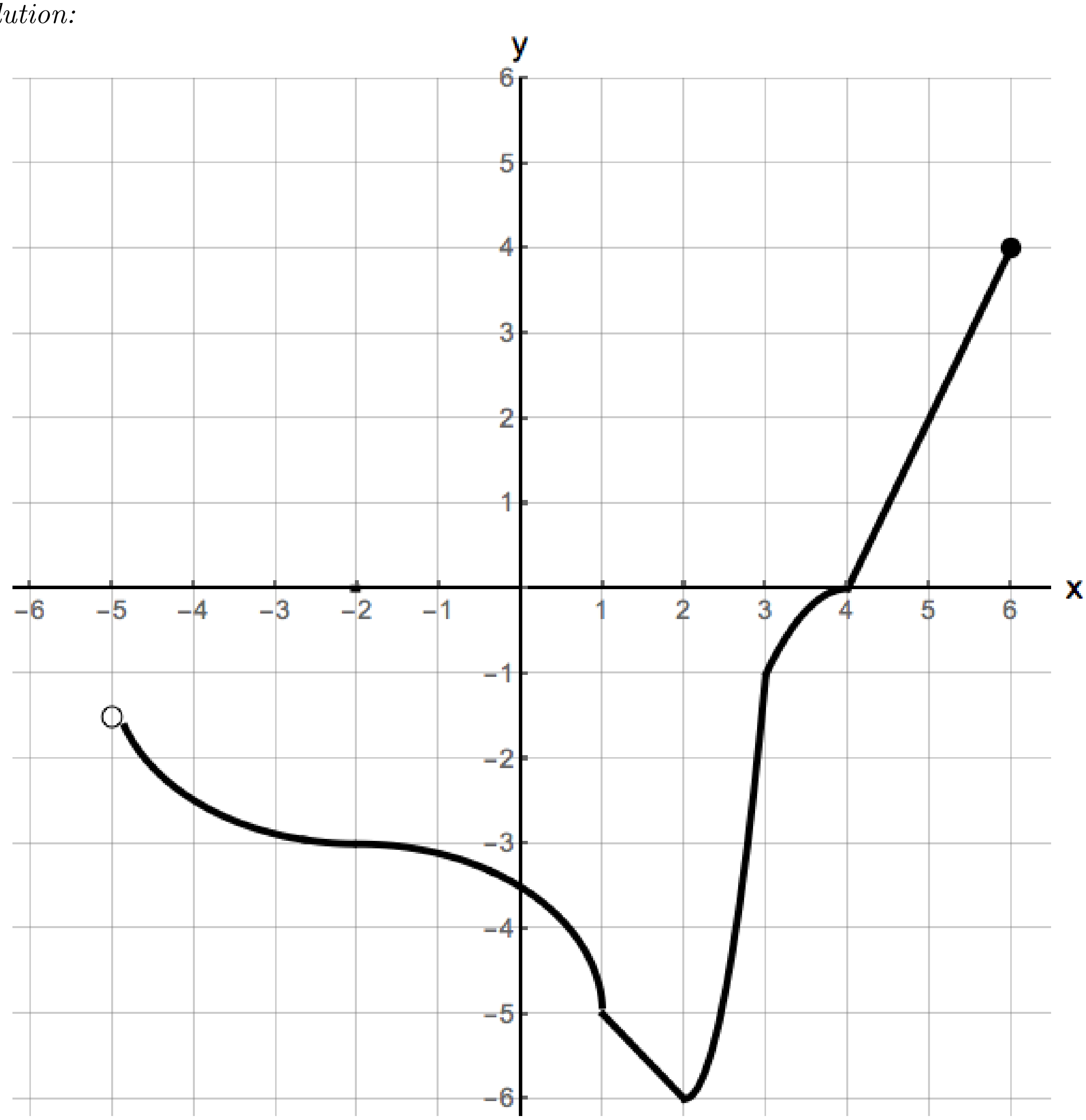
Given the values of p, W and Z shown in the tables above, which of the following statements could be true? Circle all that apply.

Solution:
p is a function of W Z is a function of W $\boxed{\text{W is a function of Z}}$ None of these

2. [8 points] On the axes provided below, sketch the graph of **one possible function** $y = f(x)$, satisfying **all** of the following requirements. Your graph should clearly show the properties listed below to receive full credit.

- ☐ The domain of f is $(-5, 6]$.
- ☐ The range of f is $[-6, 4]$.
- ☐ $f(x) < 0$ for $-5 < x < 2$.
- ☐ f is decreasing on $(-5, 2)$.
- ☐ f is concave up for $-5 < x < -2$.
- ☐ f is concave down for $-2 < x < 1$.
- ☐ $f(3) = -1$.
- ☐ f has a constant rate of change for $4 < x < 6$.

Solution:



3. [11 points] At 8:00 am, a water pump is turned on and water starts filling a swimming pool. Consider the following functions:

a) Let $F(t)$ be the number of gallons of water the pump has put into the swimming pool t minutes after 8 am.

b) Let $G(x)$ be the depth of the water in the swimming pool, in inches, when it contains x gallons of water.

Assume that all the functions defined above are invertible.

a. [6 points] Give a practical interpretation to the following mathematical expressions:

Solution:

i) $G^{-1}(30)$:

The number of gallons in the swimming pool if the depth of the water is 30 inches.

ii) $G(F(30))$:

It is the depth of the water, in inches, at 8:30 am.

b. [5 points]

i) Let D be the number of gallons of water the pump puts into the swimming pool between 8:15 am and 8:30 am. Find a mathematical expression for the constant D in terms of any the functions defined above.

Solution: $D = F(30) - F(15)$

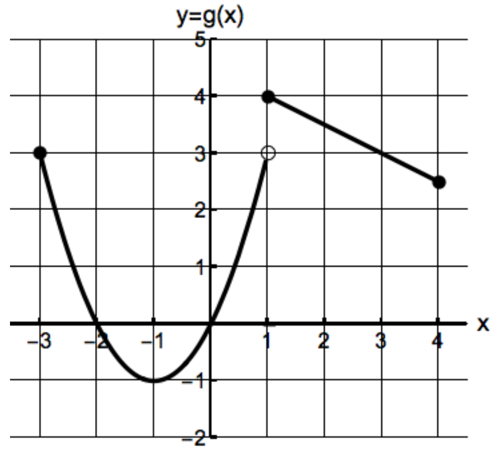
ii) Let $H(m)$ be the amount of water, in gallons, put by the water pump in the swimming pool **m minutes after 9:00 am**. Find a formula for $H(m)$ in terms of any of the functions defined above.

Solution: $H(m) = F(m + 60)$

4. [13 points] Consider the following functions

x	-2	-1	0	1	2	3
$p(x)$	0	3	2	1	-4	-2

$k(x) = x^2 - x$



a. [6 points] Find the values of the following quantities. If any of the quantities is not defined, write "UNDEFINED". Simplify your answers.

Solution:

i) $k(-4) = (-4)^4 - (-4) = 20.$

ii) $(p(2))^{-1} = (-4)^{-1} = -0.25.$

iii) $(2g(1) + 1)^2 = (2(4) + 1)^2 = 81.$

iv) $g(k(2)) = g(2) = 3.5$

b. [2 points] Find an expression for $k(t+1)$ in terms of only the variable t. You do not need to simplify it.

Solution:
$k(t + 1) = (t + 1)^2 - (t + 1)$

c. [5 points] Find all the solutions to the following equations. Your answers must be exact. If an equation has no solution, write "No solution". Show all your work.

i) $k(z) = 0.$

Solution: $z^2 - z = z(z - 1) = 0$ then $z = 0$ and $z = 1.$

ii) $p(g(t)) = -2$

Solution: $p(g(t)) = -2$ implies $g(t) = 3.$ Then $t = 3, -3.$

5. [14 points] A small country decides to reduce the amount of electrical power they produce using coal. The electrical power W generated with coal in 2008 and 2011 was 250 and 120 megawatts, respectively.

a. [8 points]

i) Suppose that $W = f(t)$, where the function f is exponential and t represents the number of years after 2004. Find a formula for $f(t)$. Your answer must be in **exact form**. Show all your work.

Solution: If $f(t) = ab^t$, then

$$ab^4 = 250 \qquad \text{and } ab^7 = 120$$

$$\frac{ab^7}{ab^4} = b^3 = \frac{12}{25} \qquad \text{then} \qquad b = \left(\frac{12}{25}\right)^{\frac{1}{3}}. \quad \text{Hence } a = \frac{250}{b^4} = \frac{250}{\left(\frac{12}{25}\right)^{\frac{4}{3}}}.$$

Therefore $f(t) = \dfrac{250}{\left(\frac{12}{25}\right)^{\frac{4}{3}}} \left(\dfrac{12}{25}\right)^{\frac{t}{3}} = 250 \left(\dfrac{12}{25}\right)^{\frac{t-4}{3}}$

ii) Find the value of the vertical intercept of the function $W = f(t)$ and give a practical interpretation of your answer. Your answer must be **exact** or include at least 2 decimals.

Solution:
Vertical intercept : $f(0) = a = \dfrac{250}{\left(\frac{12}{25}\right)^{\frac{4}{3}}}$.
Practical interpretation: The amount of electric power in megawatts generated in the small country with coal produced in 2004.

b. [6 points] A small country decides to reduce the amount of electrical power they produce using coal. The electrical power W generated with coal in 2008 and 2011 was 250 and 120 megawatts, respectively.

i) Suppose that $W = g(t)$, where the function g is linear and t represents the number of years after 2004. Find a formula for $g(t)$. Show all your work.

Solution: The slope of $g(t)$ is $m = \frac{120-250}{7-4} = -\frac{130}{3}$. Using the point-slope formula we get $g(t) = 250 - \frac{130}{3}(t-4) = \frac{1270}{3} - \frac{130}{3}t$.

ii) Find the value of $g^{-1}(0)$. Include units. Show all your work.

Solution: Let $t = g^{-1}(0)$, then $\frac{1270}{3} - \frac{130}{3}t = 0$. Solving for $t = \frac{1270}{130} \approx 9.769$ years after 2004.

6. [11 points]

a. [2 points] If the range of the function $y = H(x)$ is $(-4, 3]$, what should be the range of the function $G(x) = H(x + 10) - 20$? Write your answer using interval notation or inequalities.

Solution: $(-24, -17]$

b. [3 points] Find the domain of the function

$$k(x) = \frac{100}{\sqrt{1 - 2x}}$$

Write your answer using interval notation or inequalities. Show all your work.

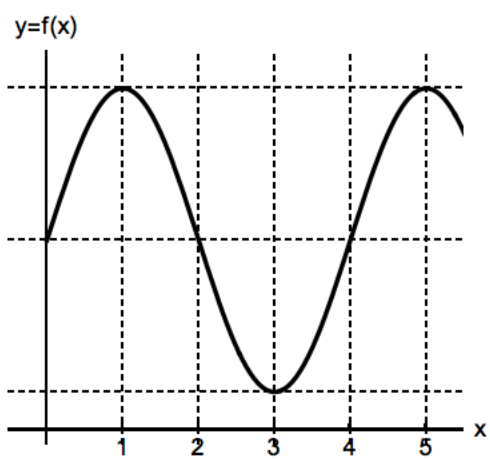
Solution: We need $1 - 2x \geq 0$ and $\sqrt{1 - 2x} \neq 0$. This implies that $1 - 2x > 0$. Hence $x < \frac{1}{2}$

c. [4 points] Find the equation of the linear function $f(x)$ that has an x-intercept at 3, and is perpendicular to the line $4x - 3y = 1$. Show all your work.

Solution: We know that $m = -\frac{3}{4}$ and $(3, 0)$ is on the graph of the line. Hence if $f(x) = -\frac{3}{4}x + b$, then $0 = -\frac{3}{4}(3) + b$. This yields $b = \frac{9}{4}$. Therefore $f(x) = -\dfrac{3}{4}x + \dfrac{9}{4}$.

d. [2 points] The graph of the function $f(x)$ is given below. In which interval is the value of the average rate of change of $f(x)$ the largest? Circle your answer.

Solution:



i) On $0 \leq x \leq 4$

ii) On $1 \leq x \leq 3$

iii) $\boxed{\text{On } 3 \leq x \leq 5}$

iv) On $2 \leq x \leq 5$

7. [7 points]

a. [3 points] A new car was sold at 35 thousand dollars. Its value depreciates 5.4 percent every year. Let $V(t)$ be the value of the car, in thousand dollars, t years after it was sold. Find a formula for $V(t)$.

Solution: $V(t) = 35(1 - 0.054)^t = 35(0.946)^t$

b. [4 points] Let $B(t)$ be the population of bats in a cave t years after 2000, where

$$B(t) = 300(1.152)^{2t}$$

1. How many bats are in the cave in 2015?

Solution: $B(15) = 300(1.152)^{30} \approx 20926.45$

2. What is is the annual percentage growth rate of this population? Your answer must be **exact** or accurate up to the first three decimals. Show all your work.

Solution: Since $b = (1.152)^2$, then $r = (1.152)^2 - 1 \approx 0.327$

8. [14 points] The owner of a restaurant has a budget to buy up to 15 hours of advertising time on the radio. She predicts that her profits $P(x)$, in thousands of dollars, when she buys x minutes of advertising on the radio for her restaurant is given by:

$$P(x) = -3x^2 + 40x + 100 \qquad \text{for} \qquad 0 \leq x \leq 15.$$

a. [5 points] Write the formula of $P(x)$ in vertex form by completing the square. Show all your work step-by-step to receive full credit.

Solution:

$$\begin{aligned} P(x) &= -3x^2 + 40x + 100 \\ &= -3\left(x^2 - \frac{40}{3}x\right) + 100 \\ &= -3\left(x^2 - \frac{40}{3}x + \left(-\frac{20}{3}\right)^2 - \left(-\frac{20}{3}\right)^2\right) + 100 \\ &= -3\left(\left(x - \frac{20}{3}\right)^2 - \left(\frac{20}{3}\right)^2\right) + 100 \\ &= -3\left(x - \frac{20}{3}\right)^2 + 3\left(\frac{20}{3}\right)^2 + 100 \\ &= -3\left(x - \frac{20}{3}\right)^2 + \frac{700}{3} \end{aligned}$$

b. [3 points] Find the practical domain and range of the function $P(x)$. Your answers must be written in **exact form** or accurate up to the first two decimals. Use inequalities or interval notation.

Solution: Domain: $[0, 15]$ Range: $\left[25, \frac{700}{3}\right]$

The statement of the problem has been included below for your convenience.

The owner of a restaurant has a budget to buy up to 15 hours of advertising time on the radio. She predicts that her profits $P(x)$, in thousands of dollars, when she buys x minutes of advertising on the radio for her restaurant is given by:

$$P(x) = -3x^2 + 40x + 100 \qquad \text{for} \qquad 0 \le x \le 15.$$

c. [3 points] What should be the minimum amount of radio advertising time the owner has to buy if she wants to obtain a profit of one hundred fifty thousand dollars?
Your answer should be obtained **algebraically** and it must be in **exact form** or accurate up to the first two decimals. Include units. Show all your work.

Solution:

$$-3x^2 + 40x + 100 = 150$$
$$-3x^2 + 40x - 50 = 0$$
$$x = \frac{-40 \pm \sqrt{(40)^2 - 4(-3)(-50)}}{-6} = \frac{40 \pm \sqrt{1000}}{6}$$
$$x = \frac{40 - \sqrt{1000}}{6} \qquad \text{hours}$$

d. [3 points] Find the average rate of change of the function $P(x)$ for $10 \le x \le 15$. Include units. Show all your work.

Solution:

$$\frac{\Delta P}{\Delta x} = \frac{P(15) - P(10)}{15 - 10} = \frac{25 - 200}{5} = -\frac{175}{5} = -35 \text{ thousands of dollars per hour}$$

9. [11 points]

a. [4 points] Susan has a budget of 150 dollars to buy coffee or green tea. The prices per pound of coffee and green tea are 8 and 40 dollars respectively. Suppose that she spends all her budget buying C pounds of coffee and G pounds of green tea. Find a formula for C in terms of G. Show all your work.

Solution: Since $8C + 40G = 150$ then $C = \dfrac{150 - 40G}{8}$.

b. [7 points] A farm sells milk to a cheese company. The farm charges two dollars per gallon and a shipping fee of 30 dollars. On orders of more than 50 gallons, the price of each gallon above the first 50 gallons is reduced to 1.80 dollars. Let $P(m)$ be the cost (in dollars) of buying m gallons of milk from the farm. Write a piecewise defined formula for $P(m)$. Your formula must reflect the practical domain of this function.

Solution:

$$P(m) = \begin{cases} 30 + 2m & 0 < m \leq 50 \\ 130 + 1.8(m - 50) = 40 + 1.8m & m > 50 \end{cases}$$