Math 105 — Final Exam

Thursday, April 21

Uniquename: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. This exam has 11 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
3. Do not separate the pages of this exam. If they do become separated, write your name on every page and point this out to your instructor when you hand in the exam.
4. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
5. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Include units in your answer where that is appropriate.
6. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. Notecards are not allowed in this exam.
7. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
8. **Turn off all cell phones and pagers**, and remove all headphones.
9. You must use the methods learned in this course to solve all problems.

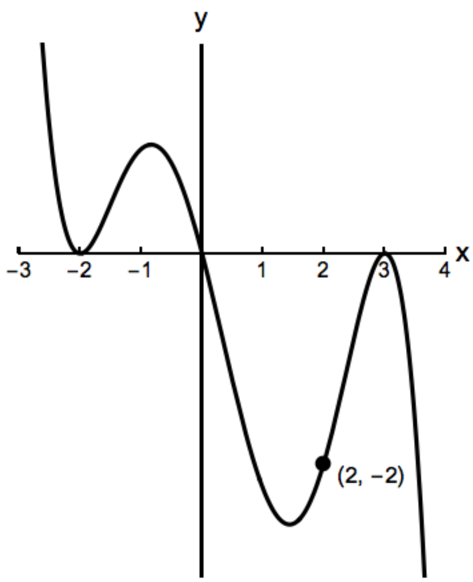
Problem	Points	Score
1	9	
2	11	
3	9	
4	13	
5	10	
6	12	
7	13	
8	9	
9	14	
Total	100	

1. [9 points]

a. [3 points] Let T be the temperature in °F at a distance L feet away from a bonfire. It is known that for $1 \le L \le 3$, the temperature T is inversely proportional to the cube root of the distance L to the bonfire. Find a formula for T in terms of L if the temperature at 2 feet away from the bonfire is 125°F.

Solution: We know that $T = \frac{k}{L^{\frac{1}{3}}}$. Since $T(2) = 125 = \frac{k}{2^{\frac{1}{3}}}$, then $k = 125\sqrt[3]{2}$. Hence $T = \frac{125\sqrt[3]{2}}{L^{\frac{1}{3}}}$.

b. [6 points] The graph of a polynomial $f(x)$ of degree five is shown below.



Solution:

i) Find the zeros of $f(x)$: $x = -2, 0$ and 3.

ii) Find a formula for $f(x)$:

Using the zeros and the graph, we can say that $f(x) = kx(x+2)^2(x-3)^2$. Since $f(2) = -2)$, then $-2 = k2(4)^2(1)^2$. This yields $k = -\frac{1}{16}$. Hence

$f(x) = -\frac{1}{16}x(x+2)^2(x-3)^2$.

2. [11 points]

a. [2 points] Let $f(x) = \dfrac{3x^2}{10x^2 + x + 1} + 5$. Find the equation of the horizontal asymptote of the graph of $f(x)$. If the graph has no horizontal asymptote, write "None".

Solution: Horizontal asymptote: $y = 5.3$.

b. [2 points] For which of the following values of x is the function $f(x) = \sin(x)$ invertible? Circle all that apply.

Solution: $\boxed{\frac{\pi}{2} \le x \le \frac{3\pi}{2}}$ $\quad 0 \le x \le \pi \quad$ $\boxed{-\frac{\pi}{2} \le x \le \frac{\pi}{2}}$ $\quad 0 \le x \le 2\pi \quad$ None of these.

c. [3 points] Find the equations of the vertical asymptotes and the x coordinate(s) of the hole(s) of the function $f(x) = \dfrac{(x-2)(x-3)}{2x^2 - 5x + 2}$.

Write "None" if the graph of this function does not have a hole or a vertical asymptote. Show all your work.

Solution: The zeros in the denominator can be found by solving $2x^2 - 5x + 2 = 0$. Using quadratic formula

$$x = \frac{5 \pm \sqrt{25 - 4(2)(2)}}{4} \qquad x = 2, 0.5.$$

Vertical asymptotes: $x = 0.5$ Holes: $x = 2$.

d. [4 points] Fill in the blanks:

i) Let $r(x) = (x^4 - 5)^4$. If with $r(x) = H(G(x))$ with $H(x) = x^4$, then

$G(x) =$

ii) Let $k(x) = 2e^{2x+1}$. If $k(x) = F(2x)$ then $F(x) =$________________.

Solution: $G(x) = x^4 - 5$ and $F(x) = 2e^{x+1}$

3. [9 points]

a. [4 points] A residential community started a paper recycling program in 2002. According to their records, the community recycled 4000 and 12000 lbs of paper in 2005 and 2013, respectively. Let $P(t)$ be the amount of paper recycled by this community (in lbs) t years after 2002. Find a formula for $P(t)$ if you assume that it is a power function. Your answer must be written in **exact** form.

Solution: Since $P(t) = kt^p$ then

$$k(3^p) = 4000 \qquad k(11^p) = 12000.$$

$$\left(\frac{11}{3}\right)^p = 3.$$

$$p\ln\left(\frac{11}{3}\right) = \ln(3) \qquad p = \frac{\ln(3)}{\ln\left(\frac{11}{3}\right)}$$

$$k = \frac{4000}{3^{\frac{\ln(3)}{\ln\left(\frac{11}{3}\right)}}} = (4000)3^{-\frac{\ln(3)}{\ln\left(\frac{11}{3}\right)}}.$$

Hence $P(t) = (4000)3^{-\frac{\ln(3)}{\ln\left(\frac{11}{3}\right)}} t^{\frac{\ln(3)}{\ln\left(\frac{11}{3}\right)}}$.

b. [5 points] Let $W(t)$ be the water consumption of the residential community, in millions of gallons, t years after 2005. The table below shows some values of $W(t)$

t	2	5	8
$W(t)$	5.38	10.51	20.52

Note: The values in the table have been rounded to the nearest 0.01.

Assume that the function $W(t)$ increases exponentially. Your answers should be written in exact form or round your answers to the nearest 0.01.

i) What is the annual percent rate of the function $W(t)$? Show all your work.

Solution: Since $W(t) = ab^t$, then $\frac{W(5)}{W(2)} = \frac{ab^5}{ab^2} = b^3 = \frac{10.51}{5.38}$.

Then $b = \sqrt[3]{\frac{10.51}{5.38}} \approx 1.25$ and $r = b - 1 = \sqrt[3]{\frac{10.51}{5.38}} - 1 \approx 0.25$.

ii) What is the annual continuous rate of $W(t)$?

Solution: $k = \ln(b) = \ln\left(\sqrt[3]{\frac{10.51}{5.38}}\right) \approx 0.22$.

4. [13 points]

a. [6 points] Two companies, Altor and Bear, decide to invest in Cease, a small start up company, in January 2014. Let $A(m)$ and $B(m)$ be the money invested in Cease, in thousands of dollars, m months after January 2014 by Altor and Bear, respectively.

i) Find a formula for $I(y)$, the amount of money, in thousands of dollars, invested by Alton and Bear on Cease y *years* after January 2014.

Solution: $A(12y) + B(12y)$

ii) Assume that only Alton and Bear invest in Cease. Find a mathematical expression that represents the fraction of the money invested in Cease by Alton in *March 2014*.

Solution: $\dfrac{A(2)}{A(2) + B(2)}$.

b. [7 points] A patient has a high fever and goes to a hospital. At the hospital, the patient receives a fever reducing medication intravenously to reduce his body temperature.

- Let $F(s)$ be the amount of medication (in milligrams) in the patient's body s minutes after the medication was administered.
- Let $G(s)$ be the patient body's temperature (in °F) s minutes after the medication was administered.

Assume that the functions F and G are invertible. Find practical interpretation of the following mathematical expressions:

i) $G(100) = 105$

Solution: The patient body's temperature is 105° F one hundred minutes after the medication was administered.

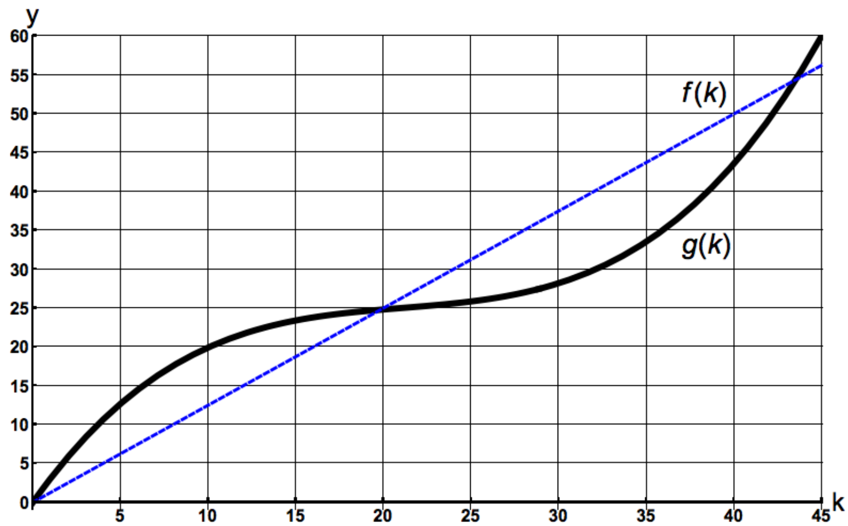
ii) $F^{-1}(100)$

Solution: The number of minutes after the medication was administered at which the patient has 100 milligrams of medication in his body.

iii) $F(G^{-1}(100))$

Solution: The amount of medication in the patient's body when his body temperature is 100°F.

5. [10 points] A company produces keychains. Let $f(k)$ be the revenue (in thousands of dollars) the company gets by selling k thousand keychains. Let $g(k)$ be the company's cost (in thousands of dollars) of manufacturing k thousand keychains. The graphs of the functions $f(k)$ (thin line) and $g(k)$ (thick line) are shown below.



Answer the following questions using the information provided in the graph above.

a. [2 points] How many keychains does the company produce if it spends 20,000 dollars manufacturing them?

Solution: 10 thousand keychains.

b. [2 points] For which values of $0 \leq k \leq 45$ is the function $g(k)$ concave down?

Solution: $0 \leq k \leq 20$.

c. [2 points] What is a practical interpretation of the slope of the function $f(k)$?

Solution: The revenue in dollars of selling each keychain.

d. [4 points] The profit obtained by the company is defined as the difference between the revenue obtained from selling keychains and the cost of producing them. Negative profits should be interpreted as losses.

i) Estimate the number of keychains the company needs to produce in order to obtain profits. Write your answer in interval notation or using inequalities.

Solution: $20 < k < 43$

ii) Estimate the amount of keychains the company needs to manufacture in order to obtain the maximum profits.

Solution: About 34 thousand keychains.

6. [12 points]

a. [2 points] Let $f(x)$ be an odd function whose domain is all real numbers except $x = 3$ and $x = -3$. Suppose that $\lim_{x\to 3^+} f(x) = \infty$ and $\lim_{x\to\infty} f(x) = -3$. Compute the following limits. Write "NI" if not enough information has been provided to answer the question.

Solution: $\lim_{x\to-\infty} f(x) = 3$ $\qquad$ $\lim_{x\to-3^-} f(x) = -\infty$

b. [2 points] Which of the following functions dominates the other functions as $x \to \infty$? Circle your answer.

Solution: $y = 20x^{500}$ $\qquad$ $y = 4(1.05)^x$ $\qquad$ $y = 1000\log(x)$ $\qquad$ $\boxed{y = 2e^{0.05x}.}$

c. [2 points] Fill in the blank space. Your answer may depend on the constant B.

If B is a constant, then $\dfrac{3^x + Bx^2}{4x^2 + Bx + 10^x} \longrightarrow \dfrac{B}{4}$ as $x \longrightarrow -\infty$.

Solution:

d. [6 points] Consider the function $y = h(x) = 2 + 3\log(4x + 10)$ with domain $x \geq 0$.

i) What is the range of $h(x)$ given that its domain is $x \geq 0$? Your answer must be written using interval notation or inequalities.

Solution: Range of $h(x)$: $[5, \infty)$

ii) Find a formula for $h^{-1}(y)$.

Solution:

$$\begin{aligned} y &= 2 + 3\log(4x+10) \\ y - 2 &= 3\log(4x+10) \\ \frac{y-2}{3} &= \log(4x+10) \\ 4x + 10 &= 10^{\frac{y-2}{3}} \\ 4x &= 10^{\frac{y-2}{3}} - 10 \\ x &= \frac{10^{\frac{y-2}{3}} - 10}{4} = h^{-1}(y). \end{aligned}$$

7. [13 points] The population of fish (in thousands) in a lake t years after 2010 is given by the function

$$F(t) = \frac{220}{1 + 2(1.35)^{-t}}.$$

a. [3 points] Find the value and give a practical interpretation of the vertical intercept of the function $F(t)$.

Solution: Vertical intercept$= F(0) = \dfrac{220}{1+2(1.35)^0} = \dfrac{220}{3} \approx 73.33$

Interpretation: There were $\frac{220}{3}$ thousand fish in the lake in 2010.

b. [4 points] When is the population in the lake equal to 150 thousand fish? Your answer must be found algebraically, written in exact form or rounded to the nearest 0.01.

Solution:

$$\begin{aligned}
\frac{220}{1 + 2(1.35)^{-t}} &= 150 \\
220 &= 150(1 + 2(1.35)^{-t}) \\
220 &= 150 + 300(1.35)^{-t} \\
300(1.35)^{-t} &= 70 \\
(1.35)^{-t} &= \frac{7}{30} \\
-t\ln(1.35) &= \ln\left(\frac{7}{30}\right) \\
t &= \frac{\ln\left(\frac{7}{30}\right)}{-\ln(1.35)} \approx 4.85 \text{ years after 2010.}
\end{aligned}$$

This problem continues on the next page.

The statement of the problem is included here for your convenience.

The population of fish (in thousands) in a lake t years after 2010 is given by the function

$$F(t) = \frac{220}{1 + 2(1.35)^{-t}}.$$

c. [3 points] Consider the graph of $y = F(t)$ for $-\infty < t < \infty$. Find the equation(s) of the horizontal asymptote(s) of the graph. If the graph has no horizontal asymptotes write "None".

Solution: The graph has horizontal asymptotes at $y = 0$ and $y = 220$.

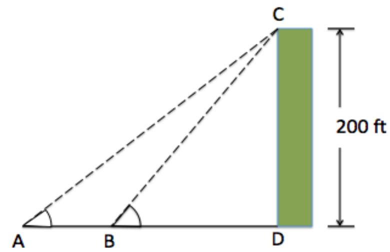
d. [3 points] Find the average rate of change of $F(t)$ for $-1 \leq t \leq 5$. Include units.

Solution:
Average rate of change of $F(t)$ for $-1 \leq t \leq 5 = \frac{F(5) - F(-1)}{6} \approx \frac{152.14 - 59.46}{6}$

≈ 15.44 thousand fish per year.

8. [9 points]

a. [5 points] Arvin and Beatrice are standing on the side of a building that is 200 feet tall at the points A and B, respectively, They are separated by a distance of 100 feet. The angle between the ground and the line BC is equal to 0.6 radians.



Note: The picture is not drawn to scale.

Your answers must be exact or rounded to the nearest 0.1.

i) Find the horizontal distance between Beatrice and the building. Show all your work.

Solution: Since $\tan(0.6) = \frac{200}{BD}$, then $BD = \frac{200}{\tan(0.6)} \approx 292.34$ feet.

ii) Let α be the angle made by the line AC and the ground. Find the value of α (in radians). Show all your work.

Solution: Since $\tan(\alpha) = \frac{200}{BD+100} \approx \frac{200}{392.34} \approx 0.509$, then $\alpha = \tan^{-1}\left(\frac{200}{392.34}\right) \approx 0.471$ radians.

b. [4 points] Let $R(t)$ be the number of employees working at a local store that specializes on selling arts and crafts t weeks after they opened for business. In their annual report, they recorded having 60 employees working for them the day the opened for business.The company had 170 and 60 employees during weeks 35 and 70 after they opened for business. Suppose that the function $R(t)$ is a quadratic function. Find a formula for $R(t)$.

Solution: The vertex of the quadratic function $R(t)$ is at $(35, 170)$. Hence $R(t) = a(t-35)^2 + 170$. Using the fact that $60 = a(-35)^2 + 170$. Then $a = -\frac{110}{(-35)^2} \approx -0.089$. Therefore $R(t) = -\frac{110}{(-35)^2}(t-35)^2 + 170$.

9. [14 points]

a. [7 points] A mass is attached to the top of a ceiling by a spring. The height of the mass above the ground oscillates from a minimum of 1.2 meters to a maximum of 2.5 meters. Let $f(t)$ be the height of the mass above the ground, in meters, at time t measured in seconds. Some of the values of the function $f(t)$ are shown below

t	0	1	2	3	4
$f(t)$	1.65	2.38	2.38	1.65	1.2

Note: All the values in the table are rounded to the nearest 0.01.
Suppose $f(t)$ is a sinusoidal function.

i) Find the period, amplitude and midline of $y = f(t)$.

Solution: Period= 5 Amplitude=0.65 Midline: $y = 1.85$

ii) Find a formula for $f(t)$.

$$f(t) = 1.85 + 0.65\cos\left(\frac{2\pi}{5}(t - 1.5)\right)$$

b. [7 points] Find all solutions to $4 - 5\sin\left(\frac{\pi}{2}x - \frac{\pi}{6}\right) = 2$ for $0 \le x \le 5$. Your answers must be found algebraically and in **exact** form.

Solution:

$$\begin{aligned}
4 - 5\sin\left(\frac{\pi}{2}x - \frac{\pi}{6}\right) &= 2 \\
5\sin\left(\frac{\pi}{2}x - \frac{\pi}{6}\right) &= -2 \\
\sin\left(\frac{\pi}{2}x - \frac{\pi}{6}\right) &= 0.4 \\
\frac{\pi}{2}x - \frac{\pi}{6} &= \sin^{-1}(0.4) \\
\frac{\pi}{2}x &= \sin^{-1}(0.4) + \frac{\pi}{6} \\
x &= \frac{2}{\pi}\left(\sin^{-1}(0.4) + \frac{\pi}{6}\right) = \frac{2}{\pi}\sin^{-1}(0.4) + \frac{1}{3}.
\end{aligned}$$

$x_1 = \frac{2}{\pi}\sin^{-1}(0.4) + \frac{1}{3}$ $\quad x_2 = \frac{7}{3} - \frac{2}{\pi}\sin^{-1}(0.4)$ $\quad x_3 = x_1 + 4$