Math 105 — First Midterm

February 9, 2017

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 12 pages including this cover. There are 10 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	21	
2	4	
3	3	
4	10	
5	13	
6	14	
7	10	
8	8	
9	9	
10	8	
Total	100	

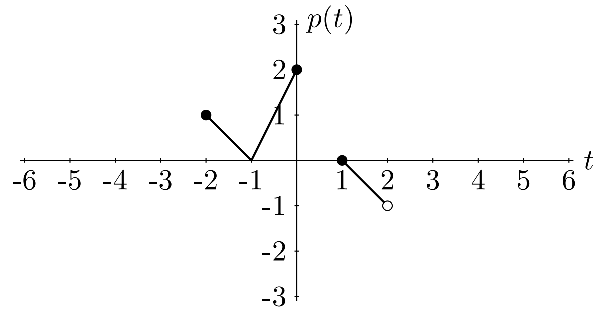
1. [21 points] Consider $h(w)$, a function with domain $[-7, 6]$, with values given in the table below.

w	-7	-5	-2	0	2	6
$h(w)$	20	10	1	-3	-5	-7

Consider the piecewise function

$$j(x) = \begin{cases} 2x + 9 & \text{for } -7 \le x < -2 \\ 20 \cdot 2^x & \text{for } -2 \le x \le 6. \end{cases}$$

Finally, consider the function $p(t)$ with graph:



a. [5 points] Circle all of the following statements that COULD be true. Circle the whole statement. Any unclear marks will be marked incorrect.

$\boxed{h(w) \text{ is invertible.}}$ $h(w)$ is concave down. $h(w)$ is exponential.

$h(w)$ is increasing. $\boxed{h(w) \text{ is decreasing.}}$ $h(w)$ is linear.

$\boxed{h(w) \text{ has two horizontal intercepts.}}$ $h(w)$ has a positive vertical intercept.

b. [4 points] Find the domain of $p(t)$ and the range of $j(x)$. Express your answer in interval notation or using inequalities.

The domain of $p(t)$ is $\underline{[-2, 0] \cup [1, 2)}$

The range of $j(x)$ is $\underline{[-5, 5) \cup [5, 20 \cdot 2^6])}$

c. [4 points] Calculate the following or write "UNDEFINED" if the quantity is not defined. Simplify your answer.

(i) $j(2) =$ $\underline{80}$

(ii) $(2p(2) - 1)^2 =$ $\underline{\text{UNDEFINED}}$

(iii) $j(h(2)) =$ $\underline{-1}$

(iv) $p(j(-4))$ $\underline{0}$

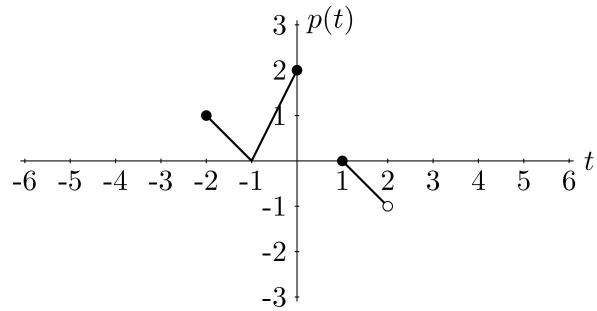
1. (continued) The information given on the previous page is given again here for your convenience:
Consider $h(w)$, a function with domain $[-7, 6]$, with values given in the table below.

w	-7	-5	-2	0	2	6
$h(w)$	20	10	1	-3	-5	-7

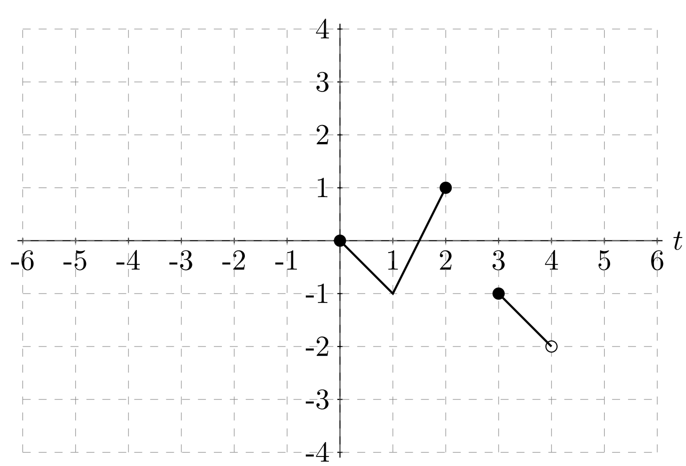
Consider the piecewise function

$$j(x) = \begin{cases} 2x + 9 & \text{for } -7 \leq x < -2 \\ 20 \cdot 2^x & \text{for } -2 \leq x \leq 6. \end{cases}$$

Finally, consider the function $p(t)$ with graph:



d. [4 points] Carefully sketch a graph of $p(t-2) - 1$ on the axes below. Be sure to make the coordinates of all endpoints of the function clear.



e. [4 points] Find all solutions to each of the equations below. Simplify your answers, but leave them in **exact** form. If an equation has no solution, write "NO SOLUTION" in the blank.

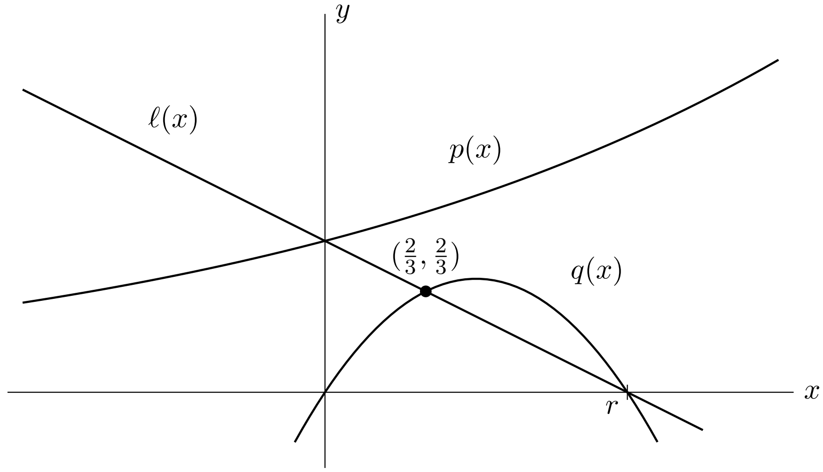
(i) $j(h(w)) = -5$.

$w =$ 6

(ii) $p(t) = 1$.

$t =$ -2, -1/2

2. [4 points] Three functions, $\ell(x)$, $q(x)$, $p(x)$ are graphed below.



These functions satisfy the following properties:

- The function $\ell(x)$ is linear with slope $-\frac{1}{2}$.
- The function $p(x)$ is exponential.
- The function $q(x)$ is quadratic with one x-intercept at $x = 0$ and the other at $x = r$.
- The graphs of $q(x)$ and $\ell(x)$ intersect once at the point $(\frac{2}{3}, \frac{2}{3})$, and again at $x = r$.

Write the correct number in each blank. Your answers should be **exact** and should not include any letters.

(i) The average rate of change of $q(x)$ between $x = \frac{2}{3}$ and $x = r$ is __-1/2__

(ii) r=__2__

(iii) $p(0) =$ __1__

(iv) $\lim_{x \to -\infty} p(x) =$ __0__

3. [3 points] The following table gives values of the variables A, B and C:

A	1	2	1	4
B	-3	0	2	1
C	7	6	5	4

Circle all of the following that could be true.

$\boxed{A \text{ is a function of } B.}$ C is a function of A.

$\boxed{B \text{ is a function of } C.}$ None of these.

4. [10 points] During the summer, Percy drives 12 miles to his uncle's farm every day. He has two scooters he can drive, scooter A and scooter B.

- Scooter A gets 35 miles per gallon in off-highway driving, and 50 miles per gallon on the highway.
- Scooter B gets 25 miles per gallon off-highway, and 55 miles per gallon on the highway.
- Two miles of the drive are off-highway, but the other 10 miles can be split any way he chooses between off-highway and highway driving.

a. [3 points] Write a function $A(h)$ that gives the number of gallons of gas Percy uses driving one way to the farm on scooter A if he drives h miles on the highway.

$A(h) = \underline{\quad \frac{h}{50} + \frac{12-h}{35} \quad}$

b. [3 points] Write a function $B(h)$ that gives the number of gallons of gas Percy uses driving one way to the farm on scooter B if he drives h miles on the highway.

$B(h) = \underline{\quad \frac{h}{55} + \frac{12-h}{25} \quad}$

c. [4 points] Which scooter should Percy drive to the farm to use the smallest amount of gas? What distance should he drive on the highway and off the highway to achieve this? How much gas does he use in one trip in which he's using the smallest amount of gas? Give your answers in **exact** form.

Percy should drive scooter __A__.

He should drive __10__ miles on the highway and __2__ miles off-highway.

He uses $\underline{\quad \frac{9}{35} \quad}$ gallons of gas on this trip.

5. [13 points] The bees on Percy's uncle's farm make honey. Over the first two weeks of May, the high temperature has been increasing each day, but the amount of honey the bees have been making decreases each day. Suppose $T(d)$ is the high temperature on the dth day of May in degrees Celsius. Let $H(d)$ be the number of gallons of honey produced by the bees on the dth day of May.

a. [3 points] Give a practical interpretation of $T^{-1}(14) = 12$ in the context of this problem.

Solution: The high temperature was 14 degrees Celsius on the 12th day of May.

b. [4 points] Give a practical interpretation of $T(H^{-1}(13)) = 10$.

Solution: On the day when 13 gallons of honey were produced, the high temperature was 10 degree Celsius.

c. [6 points] Compare the two quantities given by putting one of the symbols "<", ">", or "=" in the blank provided. If the relationship between the quantities cannot be determined, write "N" in the blank. You do not need to show your work on this problem, and there is no penalty for guessing.

$T(5)$ $\underline{\ <\ }$ $T(8)$

$H^{-1}(5)$ $\underline{\ >\ }$ $H^{-1}(8)$

$H(T^{-1}(9))$ $\underline{\ <\ }$ $H(T^{-1}(7))$

6. [14 points] After a day of work on the farm, Percy likes to toss corn cobs from the second story window of the barn to the ground. On one toss, the corn cob follows a parabolic path $h(x) = -x^2 + bx + c$ where $h(x)$ is the height of the cob above the ground, in feet, when it is a horizontal distance x feet from the barn. The numbers b and c are constants.

a. [3 points] Interpret the vertical intercept of $h(x)$ in the context of this problem.

Solution: The vertical intercept of $h(x)$ is the height of the window above the ground (in feet).

b. [4 points] If the window is 9 feet from the ground, and the cob hits the ground 9 feet from the barn, find the values of the constants b and c. Show your work.

$b =$ 8

$c =$ 9

Solution: The information in the problem tells us that $(0, 9)$ and $(9, 0)$ are both on path of the cob before the first bounce. We see $c = 9$ immediately using the point $(0, 9)$, so we just need to find b in $h(x) = -x^2 + bx + 9$. Using 9 for x and 0 for $h(x)$, $0 = -81 + 9b + 9$, so $b = 8$.

c. [4 points] After the cob bounces, it follows a path given by $p(x) = -\frac{1}{3}x^2 + 8x - 45$ where $p(x)$ is the height of the cob above the ground, in feet, when it is a horizontal distance x feet from the barn. By completing the square, find the maximum height the cob achieves after it bounces. You must show all steps of your calculation.

maximum height = 3 feet

Solution:

$$\begin{aligned} p(x) &= -\frac{1}{3}x^2 + 8x - 45 \\ &= -\frac{1}{3}(x^2 - 24x) - 45 \\ &= -\frac{1}{3}(x^2 - 24x + 144) - 45 + 48 \\ &= -\frac{1}{3}(x - 12)^2 + 3 \end{aligned}$$

So 3 feet off the ground is the maximum height achieved by the cob.

d. [3 points] Find the distance the cob is from the barn when it hits the ground for the second time. Show your work. Hint: Use the quadratic formula.

distance = 15 feet

Solution: The quadratic formula gives

$$x = \frac{-8 \pm \sqrt{64 - 4(-1/3)(-45)}}{-2/3} = \frac{-8 \pm 2}{-2/3}.$$

So $x = 9, 15$, but $x = 9$ is the location of the first bounce, so the second bounce must be at $x = 15$.

7. [10 points] Percy brought Sally to the farm one day to pick strawberries. When they first began picking, Sally was picking strawberries at a rate of 357 strawberries per hour, and she was picking strawberries at a rate of 332 strawberries per hour at the end of the second hour.

a. [4 points] Find a formula for an exponential function $R(t)$ that could model the rate at which Sally was picking strawberries t hours after they began. Give your answer in **exact** form.

$R(t) =$ $357\left(\sqrt{\frac{332}{357}}\right)^t$

Solution: Our function will be of the form $R(t) = 357b^t$ since $R(0) = 357$. Using $R(2) = 332$, we see $332 = 357b^2$. So $b = (\frac{332}{357})^{\frac{1}{2}}$.

b. [4 points] Find a formula for a linear function $L(t)$ that could model the rate at which Sally was picking strawberries t hours after they began. Give your answer in **exact** form.

$L(t) =$ $-\frac{25}{2}t + 357$

Solution: The slope of $L(t)$ is $(332 - 357)/2 = -25/2$. The vertical intercept is 357.

c. [2 points] Now assume $S(t)$ was the actual rate at which Sally was picking strawberries t hours after they began. The rate at which Percy was picking strawberries t hours after they began is given by the function $P(t) = S(t + 2)$. Which of the following is a correct practical interpretation of $P(t) = S(t + 2)$ in this context? Circle your answer.

(a) The rate at which Percy picks strawberries is equal to the rate at which Sally was picking them two hours earlier.

(b) Percy picks strawberries for two hours more that Sally.

$\boxed{\text{(c)}}$ The rate at which Percy picks strawberries is equal to the rate at which Sally will be picking them two hours later.

(d) Each hour, Percy picks two more strawberries than Sally.

8. [8 points] Percy sells tomatoes from his uncle's farm at the farmer's market. The following table shows the price $P(w)$ in dollars he charges for w pounds of tomatoes.

w	2	5	10
$P(w)$	5	10	16

a. [3 points] Find the average rate of change of $P(w)$ between $w = 5$ and $w = 10$. Include units.

The average rate of change between $w = 5$ and $w = 10$ is $\frac{16-10}{10-5} = \frac{6}{5}$ dollars per pound.

b. [3 points] Could $P(w)$ be concave up, concave down, or is neither of these possible? Write your answer in the blank provided, **and** write one sentence explaining your answer.

$P(w)$ could be concave down.

Solution: $P(w)$ could be concave down because the average rate of change appears to be decreasing (AROC is 5/3 on $[2, 5]$ and it's 6/5 on $[5, 10]$).

c. [2 points] The average rate of change of $P(w)$ between $w = 1$ and $w = 4$ is 2. Which of the following is a valid practical interpretation of this average rate of change? Circle your answer.

(i) If a customer purchases between 1 and 4 pounds of tomatoes, the cost, on average, is \$2 per pound.

(ii) Each pound of tomatoes purchased between 1 pound and 4 pounds costs \$2.

$\boxed{\text{(iii)}}$ If a customer is purchasing between 1 and 4 pounds of tomatoes, and she decides to buy a little more, she will be charged, on average, \$2 per pound for the additional amount she buys.

(iv) Four pounds of tomatoes, on average, cost \$2 more than one pound of tomatoes.

9. [9 points] For this entire problem assume:

- $f(x)$ is a decreasing function.
- $h(x)$ is a quadratic function.
- $r(x)$ is an exponential function with growth factor $\frac{1}{3}$, satisfying $r(0) > 0$.
- All of the above functions have domain $(-\infty, \infty)$.

a. [4 points] Which of the following COULD be true? Circle all that apply. Unclear answers will be marked incorrect.

$\boxed{\text{(i)}}$ The function $f(x)$ is concave up.

$\boxed{\text{(ii)}}$ The function $f(x)$ is exponential.

(iii) The function $f(x)$ is quadratic.

$\boxed{\text{(iv)}}$ The function $f(x)$ has no x-intercepts.

(v) The average rate of change of $f(x)$ between $x = 1$ and $x = 5$ is 1.

b. [3 points] Which of the following MUST be true? Circle all that apply. Unclear answers will be marked incorrect.

(i) The function $h(x)$ has at least one x-intercept.

(ii) The average rate of change of $h(x)$ between $x = 1$ and $x = 2$ is less than the average rate of change of $h(x)$ between $x = 2$ and $x = 3$.

$\boxed{\text{(iii)}}$ The average rate of change of $r(x)$ between $x = 1$ and $x = 2$ is less than the average rate of change of $r(x)$ between $x = 2$ and $x = 3$.

$\boxed{\text{(iv)}}$ $r(-2)$ is positive.

c. [2 points] Compute $\dfrac{r(100)}{r(98)}$ in **exact** form.

$$\frac{r(100)}{r(98)} = \underline{\quad 1/9 \quad}$$

10. [8 points] The shape of Percy's favorite hill on his uncle's farm can be visualized as the graph of a piecewise function $y = f(x)$. The function is quadratic on the interval $[-5, 3)$, and it's exponential on the interval $[3, 10]$. The function satisfies the following properties:

- $x = -5$ is a zero of $f(x)$.
- $f(x)$ has y-intercept 10.
- $f(2) = 7$.
- $f(3) = 4$.
- For $3 \leq x \leq 9$, when x increases by one, $f(x)$ decreases by 20%.

Write a formula for $f(x)$. Your answer will be graded based on whether it satisfies the criteria in the problem.

$$f(x) = \begin{cases} \underline{-\frac{1}{2}(x+5)(x-4)} & \text{for } \underline{-5 \leq x < 3} \\ \underline{4(0.8)^{-3}(0.8)^x} & \text{for } \underline{3 \leq x \leq 10} \end{cases}$$

Solution: The quadratic part of $f(x)$ can be written $a(x + 5)(x - r)$ since -5 is a zero. $f(0) = 10$, so $-5ar = 10$ or $-ar = 2$. We also know $f(2) = 7$, so $7 = 7a(2-r)$ or $1 = 2a - ar$. Combining these facts, we get $1 = 2a + 2$ or $a = -1/2$. This means $r = 4$.

The exponential part of $f(x)$ has growth factor 0.8 because it's decreasing by 20% for each increase in x by one, so we can write it as $a(0.8)^x$. Using $f(3) = 4$, we get $4 = a(0.8)^3$ or $a = 4(0.8)^{-3}$.