Math 105 — Final Exam

April 24, 2017

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	10	
2	14	
3	10	
4	7	
5	12	
6	9	
7	14	
8	9	
9	15	
Total	100	

1. [10 points] Be sure to show your work on this problem. Parts **a.** and **b.** are not related.

a. [4 points] Solve for the exact value(s) of w in the equation

$$\log(1-w) - \log(1+w) = 1.$$

If there there are no solutions, write "no solutions" in the blank and explain your answer.

$w =$ ___-9/11___.

Solution: Combining the logs, we have

$$\log\left(\frac{1-w}{1+w}\right) = 1.$$

Using both sides as an exponent of 10, gives $\frac{1-w}{1+w} = 10$, so $1-w = 10(1+w)$. Combining like terms gives us $-9 = 11w$, so $w = -9/11$.

b. [6 points] Write the quadratic function $y = -2x^2 + 16x - 1$ in vertex form by completing the square, write the x and y coordinates of the vertex, and indicate whether the vertex is a minimum, maximum or neither by circling the appropriate option.

In vertex form, $y =$ ___$-2(x-4)^2 + 31$___.

The vertex is $(x, y) =$ ___(4,31)___.

The vertex is a:

[maximum] minimum neither

Solution: The leading coefficient of this function is negative, so whatever our vertex is, it's a max because the parabola opens downward. To complete the square, first we factor out a -2 from the first two terms to get

$$-2(x^2 - 8x) - 1.$$

We need to add 16 inside the parentheses and so we compensate for this by adding 32 outside the parentheses

$$-2(x^2 - 8x + 16) - 1 + 32.$$

Factoring the perfect square we created and combining the constants outside the parentheses, we get

$$-2(x-4)^2 + 31.$$

2. [14 points] The following table gives values of three functions at three different x values.

x	1	4	9
$f(x)$	5	-4	-13
$g(x)$	48	6	3/16
$h(x)$	2	4	6

a. [4 points] Peter thinks $f(x)$ is **linear**. Find Peter's formula for $f(x)$ in exact form, if possible. If $f(x)$ can't be linear based on the information given, write "not possible" in the blank and explain why it can't be linear.

$f(x) =$ not possible .

Solution: This function can't be linear. The average rate of change on $[1, 4]$ is -3, but on $[4, 9]$ it's -9/5. Linear functions must have constant average rates of change, so this function is disqualified.

b. [5 points] Sarah thinks $g(x)$ is **exponential**. Find Sarah's formula for $g(x)$ in exact form, if possible. If $g(x)$ can't be exponential based on the information given, write "not possible" in the blank and explain why it can't be exponential.

$g(x) =$ $96(0.5)^x$.

Solution: If we try to write an exponential function, we can use the points $(1, 48)$ and $(4, 6)$ and the equation $g(x) = ab^x$. This gives us the system of equations $48 = ab$ and $6 = ab^4$. Eliminating a, we get $\frac{1}{8} = b^3$, so $b = 0.5$. This means $a = 96$. The function we found also passes through the third point $(9, \frac{3}{16})$.

c. [5 points] Sally thinks $h(x)$ is a **power function**. Find Sally's formula for $h(x)$ in exact form, if possible. If $h(x)$ can't be a power function based on the information given, write "not possible" in the blank and explain why it can't be a power function.

$h(x) =$ $h(x) = 2x^{1/2}$.

Solution: If we try to write a power function, we can use the points $(1, 2)$ and $(4, 4)$ and the equation $h(x) = kx^p$. The first point immediately gives us $2 = k$, and so $4 = 2(4)^p$ (using the second point). We can solve for p using logs or common sense, but either way, $p = 1/2$. The function we found also passes through the point $(9, 6)$, so we have our answer.

3. [10 points] Be sure to show your work on this problem. Parts **a.** and **b.** are not related.

a. [5 points] Consider the function $P = f(t) = 4 - e^{2t+5}$. Find $f^{-1}(P)$. Find a horizontal asymptote of $f(t)$.

$f^{-1}(P) =$ $\frac{\ln(4-P)-5}{2}$.

A horizontal asymptote of $f(t)$ is $P = 4$.

Solution: First, $f(t)$ is a shifted exponential function. Before the shift, the horizontal asymptote is $P = 0$ (all exponentials have this asymptote). But there is a reflection over the x-axis and a shift up 4 units, so the new asymptote is $P = 4$. To find the inverse, we need to solve for t, so we begin by subtracting 4 from both sides and then multiplying by -1 on both sides to get

$$4 - P = e^{2t+5}.$$

Taking ln on both sides gives us

$$\ln(4 - P) = 2t + 5.$$

Now we subtract 5 and divide by two

$$t = \frac{\ln(4 - P) - 5}{2}.$$

b. [5 points] Find all solutions x to the equation

$$4\cos(4x) = 1$$

on the interval $[-1, 2]$. Write your answer(s) in exact form. If there are no solutions, write "no solutions" in the blank and explain your answer.

$x =$ $\pm\frac{\arccos(1/4)}{4}, \frac{\pi}{2} \pm \frac{\arccos(1/4)}{4}$.

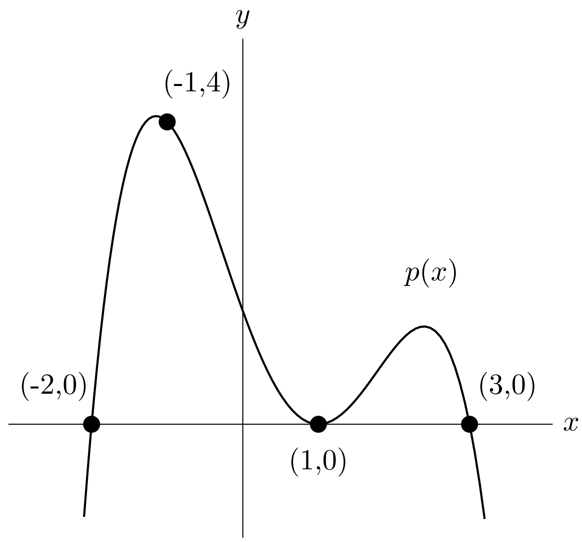
Solution: The given equation is equivalent to

$$\cos(4x) = 1/4.$$

Looking on a calculator, we can see there are 4 intersection points between $y = \cos(4x)$ and $y = 1/4$. One of these is $x = \frac{\arccos(1/4)}{4}$. Let's abbreviate this solution θ Based on the symmetry of the cosine function, $-\theta$ is also a solution. The other two solutions correspond to the first two solutions each shifted one period ($\pi/2$) to the right, so all the solutions are:

$$x = \pm\frac{\arccos(1/4)}{4}, \frac{\pi}{2} \pm \frac{\arccos(1/4)}{4}.$$

4. [7 points] Find a possible formula for the polynomial $p(x)$ whose graph is shown below. Show your work in the open space below the blank.



$p(x) =$ $\underline{\quad -0.25(x+2)(x-1)^2(x-3) \quad}$.

Solution: The function has zeros at $x = -2, 1, 3$, and appears to have an even repeated root at $x = 1$. We can tell because the function looks like the vertex of a parabola (or possibly a higher order power function with even positive exponent) near the point $(1, 0)$. This means a possible formula is

$$p(x) = a(x+2)(x-1)^2(x-3)$$

for some constant a (which should be negative based on the end behavior of the function). To find out what a is, we use the fourth point $(-1, 4)$ in our equation:

$$4 = a(-1+2)(-1-1)^2(-1-3).$$

This gives us $a = -0.25$.

5. [12 points] Percy is analyzing the cost of feeding the animals on his uncle's farm.

- Suppose $W(p)$ is the total amount of pig food, in pounds, consumed by p pigs each day.
- Suppose $C(x)$ is the cost, in dollars, of x pounds of pig food.
- Suppose there are N pigs on the farm on May 1.
- Suppose the cost of the food consumed by the **goats** on May 1 was K dollars.

a. [6 points] Write a practical interpretation of the following expressions.

$C(W(37))$ is the cost, in dollars, of the food consumed by 37 pigs in one day.

$W^{-1}(111)$ is the number of pigs that consume 111 pounds of food in one day.

b. [6 points] For each description below, write an expression using function notation and possibly the numbers K and N from above that gives the quantity described. Circle your answers.

The average amount of food, in pounds, consumed per pig on the farm on May 1.

Solution: $\boxed{W(N)/N}$

The cost of z ounces of pig food. *(Hint: There are 16 ounces in one pound.)*

Solution: $\boxed{C(z/16)}$

The total cost, in dollars, of the goat food and the pig food consumed by the animals on May 1.

Solution: $\boxed{C(W(N)) + K}$

6. [9 points] The average high temperature in Anchorage, Alaska increases from a low of 15 degrees Fahrenheit at the beginning of the 6th week of the year to a high of 61 degrees Fahrenheit at the beginning of the 32nd week. For your reference, there are 52 weeks in a year. Suppose the average high temperature in Anchorage w weeks after the beginning of the first week of the year can be modeled by a sinusoidal function $T(w)$.

a. [4 points] Find the period, amplitude and midline of the function $T(w)$.

The period is __52__.

The amplitude is __23__.

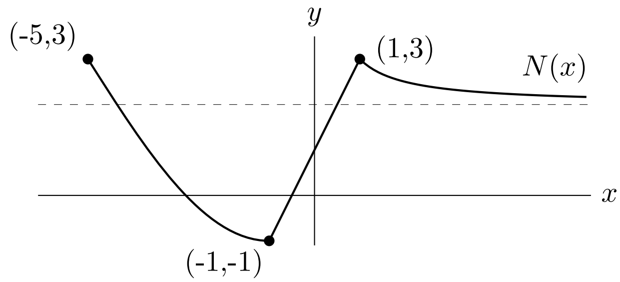
The midline is __$T = 38$ or $y = 38$__.

b. [5 points] Give a possible formula for $T(w)$. Leave all constants in exact form.

$$T(w) = -23\cos\left(\frac{\pi}{26}(w-5)\right) + 38 .$$

Solution: We know the amplitude, period and midline from part (a) so all we need is the horizontal shift to write the function completely. Since the low point is at the point $(5, 15)$, we can use cosine with a horizontal shift of 5 to the right and with a minus sign in front of the amplitude.

7. [14 points] Consider the graph of the function $N(x)$ and the formula for the function $L(t)$ represented below. $N(x)$ is linear on $[-1, 1]$, and the dotted line is a horizontal asymptote of $N(x)$ at $y = 2$. You do not need to show your work for this problem.



$$L(t) = \begin{cases} \frac{-8(t+2)(t+1)}{t^2+4} & \text{for } t < 0 \\ \frac{9(t-4)}{t^2-9} & \text{for } t \geq 0 \end{cases}$$

a. [6 points] Find the following (write "DNE" if the quantity does not exist):

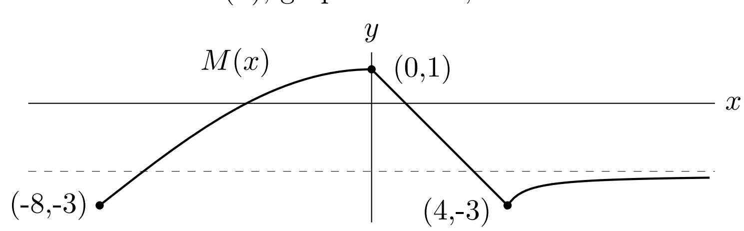
- $L(N(-1)) =$ ___0___.
- $N(L(5)) =$ ___$1 + \frac{18}{16}$___.
- $\lim_{t\to\infty} L(t) =$ ___0___.
- $\lim_{t\to-\infty} L(t) =$ ___-8___.
- $\lim_{x\to\infty} N(x) =$ ___2___.
- The average rate of change of $N(x)$ between $x = -5$ and $x = 0$ is ___-2/5___.

b. [5 points] Find all vertical asymptotes and zeros of $L(t)$. If there are none, write "none" in the corresponding blank

The vertical asymptote(s) of $L(t)$ is/are ___$t = 3$___.

The zero(s) of $L(t)$ is/are ___$t = 4, -1, -2$___.

c. [3 points] Find a formula for $M(x)$, graphed below, as a transformation of $N(x)$.



$M(x) =$ ___$-N(\frac{1}{2}(x-1))$___.

8. [9 points] The owner of Sarah and Peter's regular pizza place offers a special at dinner. When customers order a pizza, an order of breadsticks, and a large salad, they get a discount. Suppose a customer normally pays \$15 for each pizza, \$5 for an order of breadsticks, and \$8 for a large salad.

a. [3 points] The owner has been experimenting with different discounts. When he offers a discount of d dollars on the dinner special, he sells $C(d)$ specials each day. Write an expression, possibly involving $C(d)$ and d, for $R(d)$ the total revenue (sales in dollars, with no expenses) from dinner specials per day.

$R(d) = \underline{\quad (28 - d)C(d) \quad}.$

b. [2 points] The cost of all the ingredients to make each dinner special is \$8. Write an expression, possibly involving $C(d)$ and d, for $P(d)$, the total revenue from the dinner special minus the cost of the ingredients, when the discount on the dinner specials is d dollars.

$P(d) = \underline{\quad (20 - d)C(d) \quad}.$

c. [4 points] After experimenting with different discounts, the owner discovers that

$$R(d) = \frac{1}{100}d(28 - d)(78 - d).$$

Find formulas (involving only d) for $C(d)$ and $P(d)$.

$C(d) = \underline{\quad \frac{1}{100}d(78 - d) \quad}.$

$P(d) = \underline{\quad \frac{1}{100}d(20 - d)(78 - d) \quad}.$

9. [15 points] For each problem, circle the correct answer. There is only one correct answer for each part. You do not need to show your work, but unclear answers will be marked incorrect.

a. [3 points] A vertical asymptote of the function $y = \dfrac{2x^2 - x}{x - \frac{1}{2}}$ is

$x = 2$ $\qquad$ $x = \frac{1}{2}$ $\qquad$ $x = 1$ $\qquad$ $x = 3$ $\qquad$ $\boxed{\text{None of these}}$

b. [3 points] The limit of the function $y = \dfrac{3\ln x - 2\sqrt{x}}{\ln x + x}$ as $x \to \infty$ is

2 $\qquad$ 3 $\qquad$ ∞ $\qquad$ $-\infty$ $\qquad$ $\boxed{0}$ $\qquad$ None of these

c. [3 points] Suppose $k(w)$ is an odd function defined for all real numbers w, and $k(3) = -7$. If $h(w) = 2k(w + 1)$, which of the following **must** have a value of 14?

$h(-3)$ $\qquad$ $h(-2)$ $\qquad$ $h(2)$ $\qquad$ $\boxed{h(-4)}$ $\qquad$ None of these

d. [3 points] Suppose $C(t)$ is a periodic function with period 11. If $C(9) = -7$, which of the following **must** be true?

$C(t)$ has an amplitude of 5.5. $\qquad$ $C(t)$ is sinusoidal. $\qquad$ $C(-2) = 7$.

$C(14.5) = 7$. $\qquad$ $\boxed{C(-2) = -7}$ $\qquad$ None of these

e. [3 points] Suppose $P = 3$ when $Q = 7$, and $P = 2$ when $Q = -7$. Which of the following **could** be true?

Q is a decreasing function of P. $\qquad$ P is inversely proportional to Q.

$P = \dfrac{1}{49}(Q - 7)^2 + 3$. $\qquad$ $\boxed{P \text{ is an invertible function of } Q.}$ $\qquad$ None of these