Math 105 — First Midterm

February 8, 2018

UMID: EXAM SOLUTIONS Initials: ____________

Instructor: ____________ Section: ____________

---

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 9 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	11	
2	12	
3	14	
4	8	
5	10	
6	15	
7	10	
8	10	
9	10	
Total	100	

1. [11 points] Mad scientist Kiki LeBlanc is continuing her experiments with size-change technology. She is trying out her technology on ants. Below is a table showing some data for w, the weight of an ant in grams, ℓ, the length of an ant in cm, and t, the strength of an ant in marches (a unit of strength). Suppose t is a function of w.

w	0.1	0.25	1	2	2.5
ℓ	0.05	0.10	0.15	0.2	0.25
t	5	4	3	4	5

a. [3 points] Circle all statements that could be true given the information in the table. Any unclear answers will be marked incorrect.

- ℓ could be a function of t.
- [✓ t could be a function of ℓ.]
- w could be a *linear* function of ℓ.
- [✓ ℓ could be a function of w.]

b. [3 points] If the function f relates t and w, i.e. $t = f(w)$, could f be only concave up, only concave down, or is it not possible for f to be either only concave up or only concave down? Give a brief justification.

Solution: f could be only concave up because the average rates of change between consecutive values in the table are increasing.

c. [3 points] Find the average rate of change of t between $w = 0.25$ and $w = 2.5$. Leave your answer in exact form, and don't forget to include units.

The average rate of change of t between $w = 0.25$ and $w = 2.5$ is $\frac{1}{2.25}$ marches/gram .

d. [2 points] Give a practical interpretation of the rate of change you found in part (c).

Solution: Our answer from (c) means that, on average, between weights of 0.25g and 2.5g, ants gain $\frac{1}{2.25}$ of strength for each increase in weight of 1g.

2. [12 points] Consider the function $y = p(x) = 2x^2 - \sqrt{33}x - 6$.

a. [4 points] Find the zeros of $p(x)$ in exact form, if there are any, or explain why there aren't any. Show your work. Answers obtained using a calculator with no work shown will receive no credit.

The zeros of $p(x)$ are $\frac{\sqrt{33}\pm 9}{4}$

Solution: Using the quadratic formula, we have

$$x = \frac{\sqrt{33} \pm \sqrt{33 - 4(-6)(2)}}{4} = \frac{\sqrt{33} \pm 9}{4}$$

b. [5 points] Find the $x-$ and $y-$coordinates of the vertex of $p(x)$ by completing the square. You must show all your steps and write $p(x)$ in vertex form to receive credit.

The vertex of $p(x)$ is $(\frac{\sqrt{33}}{4}, -\frac{81}{8})$

Solution: $p(x) = 2x^2 - \sqrt{33}x - 6$.
$p(x) = 2(x^2 - \frac{\sqrt{33}}{2}x) - 6$.
$p(x) = 2(x^2 - \frac{\sqrt{33}}{2}x + \frac{33}{16}) - 6 - 2(\frac{33}{16})$.
$p(x) = 2(x^2 - \frac{\sqrt{33}}{4})^2 - \frac{81}{8}$.

c. [3 points] Suppose $p(x + h) = 2x^2 + \sqrt{33}x - 6$ for some number h. Find h. Support your answer with graphical or algebraic evidence.

$h =$ $\frac{\sqrt{33}}{2}$

Solution: Completing the square for $p(x + h)$ is identical to the calculation for $p(x)$ except you have $(x + \frac{\sqrt{33}}{4})^2$ instead of $(x - \frac{\sqrt{33}}{4})^2$. This means

$$p(x + h) = 2(x^2 + \frac{\sqrt{33}}{4})^2 - \frac{81}{8}.$$

This means the vertex of this new function is $(-\frac{\sqrt{33}}{4}, -\frac{81}{8})$, so the shift must have been $\frac{\sqrt{33}}{2}$ to the left.

3. [14 points] Kiki and her pet mouse Mimi (who is now the size of a small dog via size-change technology) like to go to the park and play frisbee. Suppose after t minutes of playing frisbee at the park, Mimi's satisfaction level, in pleasits (a unit of satisfaction) is given by an exponential function $M(t)$ whose values are given in the table below.

t	1	2	3	4
$M(t)$	9/2	6	8	32/3

Express all answers for all parts of this problem in **exact form**.

a. [4 points] Find the growth factor for $M(t)$ and fill in the missing values of $M(t)$ in the table.

The growth factor for $M(t)$ is $\frac{4}{3}$.

Solution: If b is the growth factor, we know $b^3M(1) = M(4)$. So $\frac{9}{2}b^3 = \frac{32}{3}$. This mean $b = \frac{4}{3}$. We can get the values in the table by starting with $\frac{9}{2}$ and multiplying by the growth factor.

b. [4 points] If Mimi never plays frisbee for more than 30 minutes, find the domain and range of $M(t)$.

The domain of $M(t)$ is $[0, 30]$.

The range of $M(t)$ is $[\frac{27}{8}, \frac{27}{8}(\frac{4}{3})^{30}]$.

Solution: We find the domain from the context of the problem. We find the range knowing that M is exponential growth so the highest point is at $t = 30$, and the lowest is at $t = 0$.

c. [3 points] Kiki's satisfaction level in pleasits, $Q(t)$, t minutes after she starts playing frisbee is an exponential function, $Q(t) = 10e^{0.02t-2}$. Is $Q(t)$ an exponential growth function or an exponential decay function? **Circle** GROWTH or DECAY in the sentence below and state the *continuous* growth or decay rate either as a decimal or as a percentage.

The *continuous* [✓ GROWTH] or DECAY rate is 0.02 or 2%.

d. [3 points] Find Kiki's satisfaction level when she first begins playing frisbee, and find the per minute (non-continuous) growth rate of her satisfaction level, $Q(t)$, either as a decimal or as a percentage.

Kiki's satisfaction level is $Q(0) = 10e^{-2}$ when she first begins playing.

The per minute growth rate of her satisfaction level is $e^{0.02} - 1$ or $100(e^{0.02} - 1)\%$.

4. [8 points] Hugo LeBlanc inadvertently introduced a rogue variety of algae in Lake Balkash at noon on June 12th. Kiki measured that the algae covered an area of 12.45 square meters on June 25th at noon, and she measured that the algae covered an area of 15.63 square meters on June 27th at noon.

a. [4 points] Assuming that the area covered by the algae grows exponentially, what was the initial area covered by the algae? Give your answer in exact form.

The algae initially covered $12.45/(\sqrt{\frac{15.63}{12.45}})^{13}$ square meters .

Solution: To find the initial amount a, we can write the equations

$$ab^{13} = 12.45$$
$$ab^{15} = 15.63$$

This means $b^2 = \frac{15.63}{12.45}$, so $b = \sqrt{\frac{15.63}{12.45}}$. We can now use either equation above to solve for a. For example, $a(\sqrt{\frac{15.63}{12.45}})^{13} = 12.45$. And then $a = 12.45/(\sqrt{\frac{15.63}{12.45}})^{13}$.

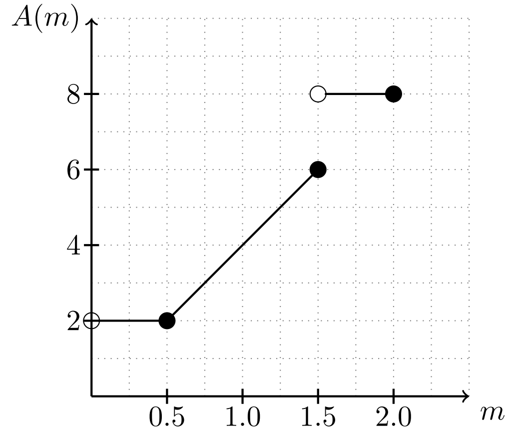
b. [4 points] Using exact form, find a formula for the area $A(t)$ that the rogue algae covers t days after Hugo introduced the algae. Be sure your formula is consistent with your answer from part (a).

$A(t) =$ $\left(12.45/(\sqrt{\frac{15.63}{12.45}})^{13}\right)\left(\sqrt{\frac{15.63}{12.45}}\right)^t$.

Solution: We did a lot of work in (a) already. We know

$$A(t) = ab^t = \left(12.45/(\sqrt{\frac{15.63}{12.45}})^{13}\right)\left(\sqrt{\frac{15.63}{12.45}}\right)^t$$

5. [10 points] Some information about three functions A, B and C is given below.



m	3	4	5	10	12
$B(m)$	5	6	6	2	6

$C(m)$ is an exponential decay function with a (non-continuous) decay rate of 70%, and vertical intercept 5.

For each part, you do not need to show your work, but you may receive partial credit for work shown.

a. [2 points] $A(C(1)) = A(1.5) = 6$

b. [2 points] $A(B(10)) = A(2) = 8$

c. [2 points] $\lim_{m\to\infty} C(m) = 0$ because C is an exponential decay function.

d. [4 points] Find all solutions to the equation

$$B(A(m)) = 6.$$

$m =$ 1, 1.25

Solution: $B(A(m)) = 6$ means $A(m) = 4$ or $A(m) = 5$. These equations give us $m = 1$ and $m = 1.25$, respectively.

6. [15 points] Kiki is beginning to experiment with time travel. She is sending her old math notebooks through time to test her machine. The machine is not working the way she intended:

- When a notebook of mass m kg is put into the machine, it travels in time $y = g(m)$ years (positive y means travel into the future, and negative y means travel into the past).
- Kiki's level of irritation while putting notebooks into her time machine, I, measured in frustrits (a unit of irritation) is a **linear** function of m, the mass, in kg, of the notebook she puts into the machine (i.e. $I = f(m)$ for some function f).

a. [6 points] Give practical interpretations of the following:

- $f^{-1}(1) = \frac{4}{3}$ means Kiki's irritation level is 1 frustrit when she puts a notebook of mass 4/3 kg into the machine.
- $g(4) = -3$ means that a 4kg notebook travels 3 years into the past when it's put into the machine.

b. [6 points] For each of the following composition of functions, give a practical interpretation of the composition or explain why the expression does not make practical sense.

- $f(g(5))$ is nonsense. The input of f (kg) is incompatible with the output of g (years).
- $f(g^{-1}(2))$ is Kiki's irritation level in frustrits when she puts a notebook into the machine and it travels into the future 2 years.

c. [3 points] If a notebook of mass 4 kg is put into the machine, Kiki's irritation level is 3 frustrits, and if a notebook of 7 kg is put into the machine, Kiki's irritation level is 8 frustrits. Using this information, find a formula for the function f.

$f(m) =$ $\frac{5}{3}(m-4)+3$

Solution: The slope is

$$\frac{8-3}{7-4} = \frac{5}{3}.$$

Using point-slope form, we have $f(m) = \frac{5}{3}(m-4)+3$.

7. [10 points] For each blank below, choose the correct answer from the options at the bottom of the page. It may be possible to use an answer more than once. Write the complete answer carefully in the blank. Assume throughout the problem that a, b, c, h, k are each **positive** constants.

a. [4 points] Suppose you have a total of c dollars to spend, and apples cost a dollars per pound, and bananas cost b dollars per pound. If you spend all your money on apples and bananas, and you buy x pounds of apples and y pounds of bananas, the relationship is linear, i.e. $y = mx + r$. Find the slope m and the $y-$intercept r in terms of a, b, and c.

$m =$ -a/b

$r =$ c/b

Solution: The total cost amount you have to spend is $c = ax + by$. We use this to solve for y:

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

b. [4 points] If a function $f(t)$ has domain $[-1, a]$ and range $[b, c]$, what are the domain and range of the function $f(t + h) - k$?

The domain of $f(t + h) - k$ is $[-1 - h, a - h]$

The range of $f(t + h) - k$ is $[b - k, c - k]$

c. [2 points] Suppose a function $g(x)$ has a horizontal asymptote $y = -k$. What is the horizontal asymptote of $g(x - a) + b$?

The horizontal asymptote of $g(x - a) + b$ is $y =$ 0

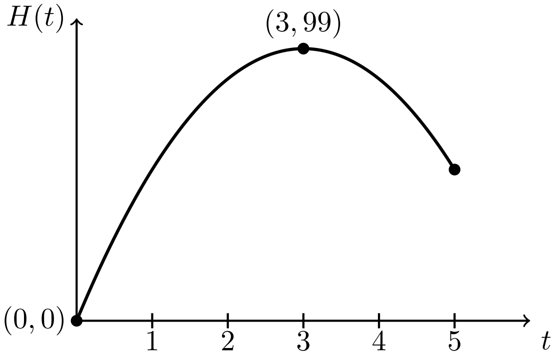
Options:

$a - h$ $a + h$ $b - h$ $b + h$ $c + h$ $c - h$ $a - k$ $a + k$

$1 - h$ $1 + h$ $h - 1$ $b + k$ $c + k$ $c - k$ $b - k$ $-1 - h$

$\frac{a}{b}$ $-\frac{a}{b}$ $\frac{b}{a}$ $-\frac{b}{a}$ $\frac{c}{a}$ $-\frac{c}{a}$ $\frac{c}{b}$ $-\frac{c}{b}$

8. [10 points] Kiki has built an jetpack that she uses to fly to her lab each day. She begins at her house and arrives at her lab 5 minutes later, reaching a maximum vertical height of 99 meters above the level of her house 3 minutes into her flight. Suppose $H(t)$, her vertical height (in meters) above the level of her house t minutes after she leaves for the lab, is a quadratic function. Assume the domain of $H(t)$ is $0 \leq t \leq 5$.



a. [3 points] On the axes above, carefully sketch graph of $H(t)$, labeling the vertical intercept and the vertex. You do not need to label the right endpoint of the graph.

b. [4 points] Find a formula for $H(t)$ based on your graph.

$H(t) =$ $-11(t-3)^2 + 99$.

Solution: Because we know the vertex of H is at $(3, 99)$, we can immediately write

$$H(t) = a(t-3)^2 + 99.$$

The. we can use the point $(0, 0)$ to get $0 = 9a + 99$. This means $a = -11$.

c. [3 points] Is Kiki's lab or house higher (vertically)? By how much? Give numerical evidence of your answer.

Solution: $H(5) = 55$ is the vertical height of the lab above Kiki's house, so the lab is higher by 55 feet.

9. [10 points] Kiki and her mother, Fifi, are restarting their failed business selling half-sized eggs that weigh half as much as regular-sized eggs. Each regular-sized egg they buy is changed into a half-sized egg via size-change technology. Customers pay six times as much per pound for the small eggs as they do for regular-sized eggs. The regular-sized eggs cost \$1 per pound (for regular customers and for Kiki and Fifi). Suppose Kiki's shrinking machine costs \$500 to build, and each shrinking machine will shrink 300 pounds of regular-sized eggs to half-sized eggs before it breaks and Kiki needs to build a new one.

a. [2 points] If N is the number of pounds of half-sized eggs they sell, how much money will they receive from the sales (in terms of N)?

They will receive $\underline{\quad 6N \quad}$ dollars from sales.

b. [3 points] Suppose the function $P = G(N)$ gives the profit, total dollars from sales minus total expenses (including all regular-sized eggs purchased, and any machines built), from selling N pounds of half-sized eggs. Find $G(5)$, $G(150)$ and $G(151)$.

$G(5) = \underline{\quad -480 \quad}$.

$G(150) = \underline{\quad 100 \quad}$.

$G(151) = \underline{\quad -396 \quad}$.

Solution: $G(5) = 6(5) - 2(5) - 500 = -480$.
$G(150) = 6(150) - 2(150) - 500 = 100$.
$G(151) = 6(151) - 2(151) - 1000 = -396$.

c. [5 points] Write a piecewise-defined formula for $G(N)$ for $0 < N \leq 400$.

$$G(N) = \begin{cases} 6N - 2N - 500 & \text{for} \quad 0 < N \leq 150. \\ 6N - 2N - 1000 & \text{for} \quad 150 < N \leq 300. \\ 6N - 2N - 1500 & \text{for} \quad 300 < N \leq 400. \end{cases}$$