Math 105 — Final Exam

April 19, 2018

UMID: EXAM SOLUTIONS Initials: ______________

Instructor: ______________ Section: ______________

1. **Do not open this exam until you are told to do so.**
2. **Do not write your name anywhere on this exam.**
3. This exam has 10 pages including this cover. There are 12 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
4. Do not separate the pages of this exam. If they do become separated, write your UMID on every page and point this out to your instructor when you hand in the exam.
5. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer questions about exam problems during the exam.
6. Show an appropriate amount of work (including appropriate explanation) for each problem so that graders can see not only your answer, but also how you obtained it. Include units in your answer where that is appropriate.
7. You may use a TI-84, TI-89, TI-Nspire or other approved calculator. However, you must show work for any calculation which we have learned how to do in this course.
8. If you use graphs or tables to find an answer, be sure to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
9. **Turn off all cell phones, pagers, and smartwatches**, and remove all headphones.

Problem	Points	Score
1	11	
2	6	
3	6	
4	6	
5	5	
6	11	
7	12	
8	5	
9	6	
10	11	
11	11	
12	10	
Total	100	

1. [11 points] The following table gives values of functions $A(t)$, $B(t)$, $B^{-1}(t)$, and $A(B(t))$ at various values of t. Assume $B(t)$ is invertible.

t	-2	0	2	3	5
$A(t)$	0	3	-2	0	2
$B(t)$	**2**	3	0	-2	5
$B^{-1}(t)$	**3**	2	-2	0	5
$A(B(t))$	-2	**0**	3	0	2

a. [3 points] Could $A(t)$ be invertible? Circle your answer and give a **brief explanation**.

YES $\boxed{\text{NO}}$

Solution: The t-values -2 and 3 both have outputs of zero, so $A(t)$ will fail the horizontal line test.

b. [3 points] Write the correct values in the three blank spaces in the table.

Solution: $B^{-1}(2) = -2$, so $B(-2) = 2$.
$B(3) = -2$, so $B^{-1}(-2) = 3$.
$A(B(0)) = A(3) = 0$.

c. [2 points] Calculate:

- $A(B^{-1}(0)) = A(2) = -2$
- $B(A(5)) = B(2) = 0$

d. [3 points] Find all solutions to the following equation that can be determined using only the information given in the table:

$$B(A(t)) = 3.$$

Solution: The input of B that outputs 3 is 0, so we set

$$A(t) = 0.$$

The table shows two inputs of A that output zero, $t = -2, 3$.

2. [6 points] After escaping from a pirate ship and being stranded at sea for several days, mad scientist Kiki LeBlanc arrived at a desert island. On the island, the temperature is very predictable, and it can be modeled by a sinusoidal function which varies daily from a high of 90°F at 4pm to a low of 64°F at 4am. Find a formula for a sinusoidal function $T(h)$ that gives the temperature in °F on the island h hours after midnight on any given day.

$$T(h) = \underline{\quad -13\cos\left(\frac{\pi}{12}(h-4)\right) + 77 \quad}$$

Solution: The midline is $T = 77$, amplitude is 13, and period is 24 (horizontal scaling is $2\pi/24$). Since $T(h)$ at its low point 4 hours after $h = 0$ (midnight), we can use a "$-\cos$" graph shifted right 4.

3. [6 points] Kiki eats lots of papayas and coconuts on the island when she's hungry. When she eats w pounds of papayas, she stays full for $P(w)$ hours. When she eats w pounds of coconuts, she stays full for $C(w)$ hours. Give practical interpretations of the following expressions:

- $C^{-1}(3) = 2$.

Solution: This means: Kiki stays full for 3 hours when she eats 2 pounds of coconuts.

- $P^{-1}(C(4))$

Solution: $P^{-1}(C(4))$ is the weight of papayas Kiki needs to eat to stay full as long as if she ate 4 pounds of coconut.

4. [6 points] A rational function $h(x)$ has zeros at $x = -1, 0, 2$, vertical asymptotes at $x = 1, 3$, and a horizontal asymptote at $y = -2$. Find a possible formula for $h(x)$. You do not need to show your work, but you may receive credit for correct work shown. There are many correct answers, and you can leave your answer unsimplified.

$h(x) =$ $\frac{-2x(x+1)(x-2)}{(x-1)(x-3)^2}$

Solution: There are lots of possible right answers. The important components are that x, $(x+1)$, $(x-2)$ appears as factors in the numerator (and not in the denominator), that $(x-1)$ and $(x-3)$ appear as factors in the denominator (and not in the numerator, or if they appear in the numerator they appear with a higher power in the denominator), that the numerator and denominator have the same degree and that the ratio of the leading coefficients is -2.

5. [5 points] The graph of the function $r(x) = \frac{x-1}{2x}$ is a transformation of the graph of the function $m(x) = \frac{1}{x}$. Fill in the following blanks with the transformations needed to transform the graph of $m(x)$ into the graph of $r(x)$. On each line use one of the phrases given below for the first blank and a number for the second blank, if applicable (for reflections, do not use the second blank). Be sure to list the transformations in the proper order. You may not need to use all four lines below, so just leave any unused lines blank.

Shift it horizontally to the right	Shift it horizontally to the left	Shift it vertically upwards	Shift it vertically downwards	Reflect it over the y-axis
Compress it horizontally	Stretch it horizontally	Compress it vertically	Stretch it vertically	Reflect it over the x-axis

To get the graph of $r(x)$ starting with the graph of $m(x)$,

first, we compress it horizontally by 1/2,

and then we reflect it over the x-axis by ____________,

and then we shift it vertically upwards by 1/2,

Solution: Note that there are many possible correct answers. The key is writing the function as $r(x) = \frac{1}{2} - \frac{1}{2x}$.

6. [11 points] Fifi has decided to use one of Kiki's time machines to travel back in time to rescue Kiki. The electrical system of the time machine is not working properly. The voltage supplied to the machine in volts t minutes after she turns it on is given by

$$y = V(t) = -130\sin(\pi(t+0.5)) + 110.$$

a. [3 points] Find the amplitude, period and midline of $V(t)$.

Amplitude: 130

Period: 2

Midline: $y = 110$

b. [3 points] To find Kiki, Fifi needs the machine to be supplied with exactly 200 volts when she travels back in time.

Find (any) one **exact** form solution to the equation

$$200 = -130\sin(\pi(t+0.5)) + 110.$$

$t =$ $\frac{1}{\pi}\arcsin(-9/13) - 0.5$

Solution: Using basic algebra,

$$\frac{-90}{130} = \sin(\pi(t+0.5)).$$

Using inverse sine and more algebra, we get

$$t = \frac{1}{\pi}\arcsin(-9/13) - 0.5.$$

c. [5 points] Using your answer from the previous part, find all times in the first three minutes after she turns on the machine when the machine is supplied with 200 volts. Show all your work and give your answers in **exact** form. No credit will be given for decimal approximations.

The times in the first three minutes when the machine is supplied with 200 volts are

$t =$ $\theta,\ \theta+2,\ 2-\theta$ minutes

Solution: If we call the solution from the previous part θ, then θ is one of the times we are looking for. There are two others (we can see this by graphing the line $y = 200$ and the graph of $V(t)$ for t-values from 0 to 3). One is a shift to the right by a period:

$$t = \theta + 2.$$

The other can be found using the symmetry of the graph of $V(t)$:

$$t = 2 - \theta \text{ or } \theta + 2(1-\theta)$$

depending on how you view the symmetry.

7. [12 points] For each question below, circle all correct answers. There could be more than one correct answer for each question. Unclear answers will be marked incorrect.

a. [2 points] If A and B are positive constants, then $\lim_{t\to\infty}(A - Be^{-t}) =$

$\boxed{\text{A}}$ $-B$ $A - B$ B 0 none of these

b. [2 points] If $y = f(x)$ has a vertical asymptote at $x = -2$, then $y = 2f(5(x+1)) - 3$ has a vertical asymptote at

-15 $-\frac{1}{5}$ -7 -4 $-\frac{3}{5}$ $\boxed{\text{none of these}}$

c. [2 points] The function $y = 3\cos(2x)$

is odd $\boxed{\text{is even}}$ $\boxed{\text{has period } \pi}$ has period 2

is not periodic is invertible has none of the attributes listed

d. [2 points] If a right triangle has an angle of 55 degrees and the side opposite that angle has length 4, the hypotenuse has length

$4\sin(35°)$ $\frac{4}{\sin(35°)}$ $4\sin(55°)$ $\boxed{\dfrac{4}{\cos(35°)}}$

$\boxed{\dfrac{4}{\sin(55°)}}$ $4\sin(35°)$ none of these

e. [2 points] Which of the following functions dominate $x^4 - 3000x$ as $x \to \infty$?

$\boxed{\left(\dfrac{9}{8}\right)^x}$ $\boxed{x^5}$ $100\log(x)$

$3000(\ln(2))^x$ $5000x^2$ none of these

f. [2 points] Which of the following functions are dominated by $x^4 - 3000x$ as $x \to \infty$?

$\left(\frac{9}{8}\right)^x$ x^5 $\boxed{100\log(x)}$

$\boxed{3000(\ln(2))^x}$ $\boxed{5000x^2}$ none of these

8. [5 points] Fifi found Kiki on the desert island after traveling back in time, and she brought another time machine with her to return both of them to their own time. Before they traveled back to the present, they ate dinner on the island, and there was a large insect flying around their table. The speed of the insect in meters per second, was **inversely proportional** to the number of minutes after they began the meal. If the insect was flying at ten meters per second two minutes after they began eating, how fast was it flying five minutes after they began eating? Be sure to show your work. Answers with no work shown will not receive credit.

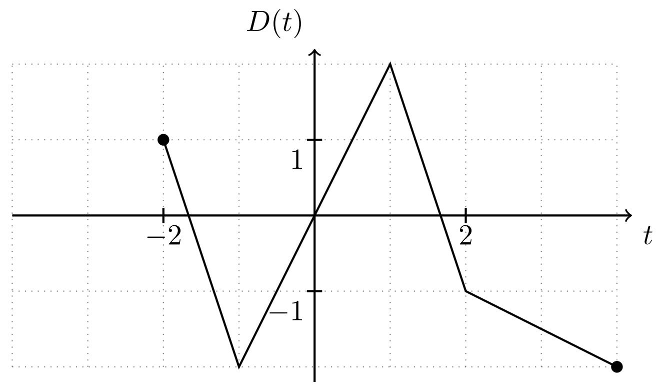
Five minutes after they began eating, the fly was flying 4 m/s

Solution: If the speed of the insect is v and the number of minutes since they began the meal is m, then the relationship described in the problem is

$$v = \frac{k}{m}.$$

Using $v = 10$ and $m = 2$, we can find $k = 20$. Then when $m = 5$, $v = 4$.

9. [6 points] The figure below shows **part** of the graph of an odd, piecewise linear, periodic function $D(t)$ of period 10, defined for all t. Use the graph to answer the questions below.



a. [2 points] Find the average rate of change of $D(t)$ between $t = -2$ and $t = 1$.

The average rate of change between $t = -2$ and $t = 1$ is (2-1)/(1-(-2))=$\frac{1}{3}$

b. [4 points] Find:

- $D(18) = D(-2) = 1$

- $D(-3) = -D(3) = 1.5$

10. [11 points] After traveling back to present day, Kiki has given up on building time travel machines, but she is still building size-change machines and testing them out on her math notebooks each weighing 1kg. She has three machines with settings ranging from 1 to 100 (including non-whole number settings). On a setting of 8, each of the three machines changes the mass of a notebook to 5kg.

a. [3 points] On a setting of 38, the first machine changes the mass of the notebook to 3.5kg. Find a formula for $L(n)$, the mass of a notebook after being transformed by the first machine on a setting of n, if $L(n)$ is a **linear** function.

$L(n) =$ $\frac{-1}{20}(n-8)+5$

Solution: The slope is $(3.5-5)/(38-8) = \frac{-1}{20}$. We can then use point slope form and the point (8,5) to get the answer.

b. [4 points] On a setting of 10, the second machine changes the mass of the notebook to $\frac{20}{9}$kg. Find a formula for $E(n)$, the mass of a notebook after being transformed by the second machine on a setting of n, if $E(n)$ is an **exponential** function.

$E(n) =$ $\frac{5}{(2/3)^8}\left(\frac{2}{3}\right)^n$

Solution: If we use the form $E(n) = ab^n$, we can set up the equations

$$\frac{20}{9} = ab^{10}$$

and

$$5 = ab^8.$$

Dividing the first equation by the second, we get $\frac{4}{9} = b^2$, so $b = \frac{2}{3}$ (growth factor must be positive). Then using the second equation above, we get $a = \frac{5}{(2/3)^8}$.

c. [4 points] On a setting of 64, the third machine changes the mass of the notebook to $\frac{5}{4}$kg. Find a formula for $W(n)$, the mass of a notebook after being transformed by the third machine on a setting of n, if $W(n)$ is a **power** function.

$W(n) =$ $20n^{-2/3}$

Solution: If we use the form $W(n) = kn^p$, we can set up the equations

$$\frac{5}{4} = k64^p$$

and

$$5 = k8^p.$$

Dividing the first equation by the second, we get $\frac{1}{4} = 8^p$, so $p = -\frac{2}{3}$. Then using the first equation above, we get $k = 20$.

11. [11 points] The two parts of this problem are **unrelated**.

a. [6 points] Consider the quadratic function $y = f(x) = -3x^2 - x + 7$. By completing the square, find both coordinates of the vertex of this parabola in exact form. Show all steps of your calculation. Circle one of the options below the blank to indicate whether the vertex is a minimum or a maximum of the function.

The vertex is $\left(\frac{-1}{6}, \frac{85}{12}\right)$

and it's a:

$\boxed{\text{MAXIMUM}}$ MINIMUM

Solution: We complete the square:

$$\begin{aligned} y &= -3x^2 - x + 7 \\ &= -3\left(x^2 + \frac{1}{3}x\right) + 7 \\ &= -3\left(x^2 + \frac{1}{3}x + \frac{1}{36}\right) + 7 + \frac{1}{12} \\ &= -3\left(x + \frac{1}{6}\right)^2 + \frac{85}{12}. \end{aligned}$$

So the maximum is $\left(\frac{-1}{6}, \frac{85}{12}\right)$

b. [5 points] Consider the function $j(t) = 4 - 2e^{-t}$. Showing your work, find $j^{-1}(P)$ if it exists or explain why it does not exist.

$j^{-1}(P) = -\ln\left(\frac{P-4}{-2}\right)$

Solution: This is a transformation of an exponential function, so it will pass the horizontal line test! To find the inverse we do the computation:

$$P = 4 - 2e^{-t}.$$

$$\frac{P-4}{-2} = e^{-t}.$$

$$-\ln\left(\frac{P-4}{-2}\right) = t.$$

12. [10 points] Consider the rational function below where n is a **positive whole** number.

$$Q(x) = \frac{(3x-1)(x+1)^2(x-2)}{(x+1)^n(x-3)}.$$

For each blank below, choose the best possible answer from the bottom of the page. There is only one best answer for each blank.

a. [2 points] $Q(x)$ has a hole at $x = -1$ for $n = 1, 2$.

b. [2 points] $Q(x)$ has a vertical asymptote at $x = -1$ for $n \geq 3$.

c. [2 points] $Q(x)$ has no horizontal asymptotes for $n = 1, 2$.

d. [2 points] $Q(x)$ has a horizontal asymptote at $y = 0$ for $n \geq 4$.

e. [2 points] $Q(x)$ has a vertical asymptote at $x = \frac{1}{3}$ for no possible values of n.

Possible answers:

for any possible value of n for no possible values of n

for $n \geq 2$ for $n \geq 3$ for $n \geq 4$ for $n = 1, 2$ for $n = 1, 2, 3$

for $n = 1, 2, 3, 4$ for $n = 2, 3$ for $n = 2, 3, 4$ for $n = 3, 4$

for $n = 1$ only   for $n = 2$ only   for $n = 3$ only   for $n = 4$ only