Math 105 — First Midterm — March 10, 2021

EXAM SOLUTIONS

---

1. This exam has 12 pages including this cover. There are 6 problems. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Please read the instructions for each individual problem carefully. One of the skills being tested on this exam is your ability to interpret mathematical questions, so instructors will not answer mathematical questions about exam problems during the exam.
3. Show an appropriate amount of work (including appropriate explanation) for each problem, so that graders can see not only your answer but how you obtained it. Partial credit will be awarded for correct work.
4. Problems may ask for answers in *exact form*. Recall that $x = \frac{1}{3}$ is an exact answer to the equation $3x = 1$, but $x = 0.333$ is not.
5. You do not need to “simplify” your answers unless asked to do so.
6. You must use the methods learned in this course to solve all problems. Logarithm functions taught in this course include “log” (log base 10) and “ln” (natural log).
7. You may use one pre-written page of notes, on an 8.5”x11” standard sheet of paper, with whatever you want written on both sides.
8. You will not be allowed to use any other resources, including calculators, other notes, or the book.
9. You must write your work and answers on **blank, white, physical paper**.
10. You must write your **initials and UMID**, but not your name or uniqname, in the upper right corner of every page of work. Make sure that it is visible in all scans or images you submit.
11. Make sure that all pages of work have the relevant problem number clearly identified.

Problem	Points
1	2
2	8
3	11
4	9
5	12
6	15
Total	57

1. [2 points] **There <u>is</u> work to submit for this problem. Read it carefully.**

- You may use your one pre-written page of notes, on an 8.5" by 11" standard sheet of paper, with whatever you want written on both sides.
- You are <u>not</u> allowed to use any other resources, including calculators, other notes, or the book.
- You may <u>not</u> use any electronic device or the internet, except to access the Zoom meeting for the exam, to access the exam file itself, to submit your work, or to report technological problems via the Google forms we will provide to do so. The one exception is that you may use headphones (e.g. for white noise) if you prefer, though please note that you need to be able to hear when the end of the exam is called in the Zoom meeting.
- You may <u>not</u> use help from any other individuals (other students, tutors, online help forums, etc.), and may <u>not</u> communicate with any other person other about the exam until **8am on Thursday** (Ann Arbor time).
- The one exception to the above policy is that you may contact the proctors in your exam room via the chat in Zoom if needed.
- Violation of any of the policies above may result in a score of zero for the exam, and, depending on the violation, may result in a failing grade in the course.

As your submission for this problem, you must write "I agree," and write your initials and UMID number to signify that you understand and agree to this policy. By doing this you are attesting that you have not violated this policy.

2. [8 points] Consider the following table of values for x, A, B, and C.

x	2	4	8	10
A	29	24	14	9
B	15.6	18.2	28.2	18.2
C	0	10	0	-3

For each of the following, decide whether the statement **could be true**. Briefly explain your reasoning.

a. B is a function of A.

Solution:
True: Since the A-values in the table don't repeat, any of the other variables is a function of A. In particular, B is a function of A.

b. A is a function of C.

Solution: **False:** There is an input ($C = 0$) which corresponds to more than one output ($A = 29$ and $A = 14$), and so A cannot be a function of C.

c. A is a linear function of x.

Solution: **True:** As in part (a) above, since the A-values in the table don't repeat, A is a function of x. Moreover, note that the average rate of change $\frac{\Delta A}{\Delta x}$ is equal to $-\frac{5}{2}$ on all intervals in the table:

$$[2,4]: \qquad \frac{\Delta A}{\Delta x} = \frac{24-29}{4-2} = -\frac{5}{2}$$

$$[4,8]: \qquad \frac{\Delta A}{\Delta x} = \frac{14-24}{8-4} = -\frac{5}{2}$$

$$[8,10]: \qquad \frac{\Delta A}{\Delta x} = \frac{9-14}{10-8} = -\frac{5}{2}$$

Therefore, A could be a linear function of x.

d. If $C = f(x)$, then $f(x)$ is concave down.

Solution:
False: $f(x)$ has average rate of change

$$\frac{\Delta C}{\Delta x} = \frac{0-10}{8-4} = -\frac{5}{2}$$

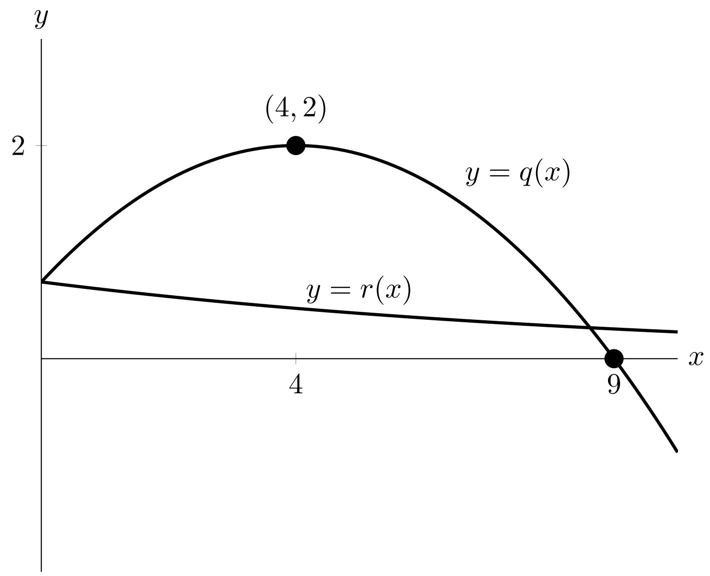
on the interval $[4,8]$, but $f(x)$ has average rate of change

$$\frac{\Delta C}{\Delta x} = \frac{-3-0}{10-8} = -\frac{3}{2}$$

on the interval $[8,10]$. Since $-\frac{5}{2} < -\frac{3}{2}$, $\frac{\Delta C}{\Delta x}$ is not decreasing, and so $f(x)$ cannot be concave-down.

3. [11 points] The graph below shows **part** of

- a quadratic function $q(x)$ with vertex and one zero marked
- an exponential function $r(x) = ab^x$ that intersects $q(x)$ on the y-axis.



a. [4 points] Find a formula for $q(x)$.

Since $q(x)$ has vertex $(4, 2)$, it has a formula in vertex form of the form

$$q(x) = k(x - 4)^2 + 2$$

for some non-zero real number k (which is the leading coefficient). We also know that 9 is a zero of $q(x)$; that is, $q(9) = 0$. We can use this equation to solve for k:

$$k(9 - 4)^2 + 2 = 0$$
$$k \cdot 25 = -2$$
$$k = -\frac{2}{25}$$

So $q(x)$ has formula $\boxed{q(x) = -\frac{2}{25}(x - 4)^2 + 2}$.

b. [2 points] What is the x-coordinate of the other zero of $q(x)$?

Solution:
Recall that the vertex $(4, 2)$ lies on the vertical axis of symmetry $x = 4$ of the graph of $q(x)$. Since one zero $x = 9$ lies 5 units to the right of this axis, the other zero lies 5 units to the left of this axis. Thus $\boxed{x = -1}$ is the other zero of $q(x)$. We can confirm this by computing $q(-1)$ from our formula.
Alternatively, you can also solve the equation $q(x) = 0$, possibly by converting to a different form.

Recall that the formula for $r(x)$ is $r(x) = ab^x$. Use the graph and your formula for $q(x)$ to answer the following questions.

c. [3 points] Which of the options below **could** be true? Briefly explain your answer.

$a < 0$ $\quad\quad$ $0 < a < 1$ $\quad\quad$ $a > 1$

Solution:
Note that the graphs of $q(x)$ and $r(x)$ intersect along the y-axis. This implies that $q(0) = r(0)$. We can use this equation to solve for a:

$$a = ab^0 = r(0) = q(0) = -\frac{2}{25}(0-4)^2 + 2 = -\frac{32}{25} + \frac{50}{25} = \frac{18}{25}.$$

Thus $\boxed{0 < a < 1}$.

d. [2 points] Which of the options below **could** be true? Briefly explain your answer.

$b < 0$ $\qquad$ $0 < b < 1$ $\qquad$ $b > 1$

Solution:
Since $r(x)$ has positive initial value and is decreasing, it is exhibiting exponential decay, and so its growth factor b satisfies the inequality $\boxed{0 < b < 1}$.

4. [9 points] An ice cream shop along the Huron river in Ann Arbor is only open in the summer. Its owner has designed a model that predicts the revenue (that is, the amount of money the shop takes in) of the shop in thousands of dollars, P, on a day where the maximum temperature is T degrees Fahrenheit.The model is described by the function $P = g(T)$, and has an inverse, $g^{-1}(P)$.

The maximum temperature in Ann Arbor, in degrees Fahrenheit, on the d^{th} day that the shop is open for the summer, is given by the function $M(d)$.

For each of the following, either give a practical interpretation of the given expression, or explain why the expression doesn't make sense in the context of the problem.

a. [3 points] $g(M(13)) = 8$

Solution:

$M(13)$ is the maximum temperature (measured in degrees Fahrenheit) on the 13th day that the ice cream shop is open. $g(M(13))$ is the ice cream shop's revenue (measured in thousands of dollars) predicted by the model on that day. Therefore, the equation $g(M(13)) = 8$ has the following interpretation:

The model predicts that the ice cream shop will take in $8 thousand on the 13th day that it is open.

b. [3 points] $g^{-1}(5)$

Solution:

$g^{-1}(5)$ is the input to g whose output corresponds to 5. The function $P = g(T)$ takes as input a daily maximum temperature (measured in degrees Fahrenheit) and returns as output the revenue (measured in thousands of dollars) of the ice cream shop predicted by the model. Therefore, the expression $g^{-1}(5)$ has the following interpretation:

the daily maximum temperature (measured in degrees Fahrenheit) at which the ice cream shop is predicted to take in $5 thousand

c. [3 points] $M(g^{-1}(7))$

Solution:

For similar reasons as above, $g^{-1}(7)$ is a temperature measured in degrees Fahrenheit. Since the inputs to the function $M(d)$ are measured in days, not degrees Fahrenheit:

It does not make sense to evaluate $M\left(g^{-1}(7)\right)$.

5. [12 points] Jack is starting a business teaching others to paint. He has come up with the following pricing plan.

- For each lesson, a client has to pay a flat fee of \$6 to cover the cost of the art supplies they will use.
- He charges \$2 per minute for the first 60 minutes of the lesson.
- He charges \$0.50 per minute for each minute after that.
- Each lesson lasts at most 120 minutes.

Let $C(m)$ be the amount of money he charges for a lesson that is m minutes long.

a. [2 points] Evaluate $C(70)$.

Solution:
$C(70)$ is the amount of money (measured in dollars) that Jack charges for a lesson that is 70 minutes long. For such a lesson, Jack charges a flat fee of \$6 to cover the cost of art supplies, \$2 per minute for the first 60 minutes of the lesson, and \$0.50 per minute for the final ten minutes of the lesson. Therefore,

$$\boxed{C(70) = 6 + 2 \cdot 60 + 0.50 \cdot 10 = 6 + 120 + 5 = 131}$$

That is, Jack charges \$131 for a 70 minute lesson.

b. [6 points] Find a formula for $C(m)$. Use standard piecewise function notation:

$$C(m) = \begin{cases} \\ \\ \end{cases}$$

Solution:
Since a lesson can last from 0 to 120 minutes long, the domain of the function $C(m)$ is given by the inequality $0 < m \leq 120$. Since Jack charges different rates for the first 60 minutes of a lesson and any remaining time afterwards, we will split this domain into two pieces: $0 < m \leq 60$, and $60 < m \leq 120$.
If a lesson is $0 < m \leq 60$ minutes long, then Jack charges a flat fee of \$6 to cover the cost of art supplies as well as \$2 per minute for all m minutes. Thus for $0 < m \leq 60$,

$$C(m) = 6 + 2m.$$

On the other hand, if the lesson is $60 < m \leq 120$ minutes long, then Jack charges a flat fee of \$6 to cover the cost of art supplies, \$2 per minute for the first 60 minutes, and \$0.50 per minute for the remaining $m - 60$ minutes. Thus

$$C(m) = 6 + 2 \cdot 60 + 0.50(m - 60) = 96 + 0.50m.$$

This gives the following piecewise-defined formula for $C(m)$:

$$\boxed{C(m) = \begin{cases} 6 + 2m, & \text{if } 0 < m \leq 60 \\ 96 + 0.50m, & \text{if } 60 < m \leq 120. \end{cases}}$$

c. [4 points] The function $d = C(m)$, where d is the cost (in dollars) of a painting lesson that lasts m minutes, is invertible. Write a formula for its inverse $C^{-1}(d)$ using standard piecewise function notation.

Solution:
In order to find a formula for the inverse of $C(m)$, we must invert the formulas given for $C(m)$ above. Since these are linear functions, this can be done algebraically as follows:

$$\begin{aligned} d &= 6 + 2m \\ 2m &= d - 6 \\ m &= 0.5d - 3 \end{aligned} \qquad\qquad \begin{aligned} d &= 96 + 0.50m \\ 0.50m &= d - 96 \\ m &= 2d - 192 \end{aligned}$$

We must also find the domains on which these formulas are valid. In order to do this, remember that inverting a function switches its domain and range. The same thing is true for the pieces of a piecewise defined function.
The formula $C(m) = 6 + 2m$ is valid on the domain $0 < m \leq 60$. On this domain, this formula has range $6 < d \leq 126$. Thus the formula $C^{-1}(d) = 0.5d - 3$ is valid on the interval $6 < d \leq 126$.
Similarly, the formula $C(d) = 96 + 0.50m$ is valid on the domain $60 < m \leq 120$. On this domain, this formula has range $126 < d \leq 156$. Thus the formula $C^{-1}(d) = 2d - 192$ is valid on the interval $126 < d \leq 156$. In summary, the inverse function $m = C^{-1}(d)$ has piecewise-defined formula

$$\boxed{C^{-1}(d) = \begin{cases} 0.5d - 3, & \text{if } 6 < d \leq 126 \\ 2d - 192, & \text{if } 126 < d \leq 156. \end{cases}}$$

6. [15 points] Scientists discover a new island in Lake Michigan and begin studying its animals. The island has both lizards and crows when they arrive, and they accidentally leave some mice on the island after discovering it.

- 5 thousand lizards live on the island when they discover it, but the population is decreasing at a rate of 5% per year.
- Half a year after the island is discovered, the population of mice has grown to 2.3 times the initial population, and appears to be growing exponentially.
- The population of crows, in thousands, t years after the island is discovered, can be modeled by $C(t) = 4e^{0.06t-1}$.

In the following problems, leave your answer in **exact form** and show every step of your work.

a. [3 points] Find a formula for $L(t)$, the number of lizards on the island, in thousands, t years after the island is discovered.

Solution:
$L(t)$ is the number of lizards on the island (measured in thousands) t years after the island is discovered. $L(t)$ is decaying exponentially with an initial value of 5 and a yearly decay rate of 5%. Thus $L(t)$ has formula

$$\boxed{L(t) = 5(1-0.05)^t}$$

b. [3 points] How long does it take for the population of mice to reach 10 times the initial population?

Solution:
Let $M(t)$ denote the number of mice living on the island t years after the island is discovered. As an exponential function, we can write $M(t) = ab^t$ for some real numbers a and b. With the information given, we cannot hope to find the initial value a, but we can find the growth factor b.
Specifically, in half a year, the population of mice has grown exponentially to 2.3 times its original size. This gives an equation $M(0.5) = 2.3 \cdot M(0)$, which we can solve to find the growth factor b:

$$\begin{aligned} ab^{0.5} &= 2.3 \cdot ab^0 \\ a\sqrt{b} &= 2.3a \\ \sqrt{b} &= 2.3 \\ b &= (2.3)^2 \end{aligned}$$

Thus $M(t) = a(2.3)^{2t}$. We now want to find how long it will take for the population of mice to reach 10 times its original size. This condition is represented by the equation $M(t) = 10 \cdot M(0)$, which we can solve for t:

$$\begin{aligned} a(2.3)^{2t} &= 10 \cdot a(2.3)^{2\cdot 0} \\ (2.3)^{2t} &= 10 \\ \log\left(2.3^{2t}\right) &= \log(10) \\ 2t \cdot \log(2.3) &= 1 \\ t &= \frac{1}{2\log(2.3)} \end{aligned}$$

Therefore, it will take $\frac{1}{2\log(2.3)}$ years for the population of mice to reach 10 times its initial size.

c. [2 points] What is the vertical intercept of $C(t)$? Interpret the meaning of this number in the context of the problem.

Solution:
The population of crows (measured in thousands) t years after the island is discovered is given by the formula $C(t) = 4e^{0.06t-1}$. This formula for $C(t)$ is not in standard or continuous growth form, so we cannot immediately read off the vertical intercept from the formula. Instead, we must compute

$$C(0) = 4e^{0.06\cdot 0-1} = 4e^{-1} = \frac{4}{e}.$$

Thus $C(t)$ has vertical intercept $\boxed{\left(0, \frac{4}{e}\right)}$.
This tells us that the number of crows when the island is discovered is $\frac{4}{e}$ thousand.

d. [2 points] By what percentage does the population of crows increase in a year?

Solution:
Let's put the formula $C(t) = 4e^{0.06t-1}$ in standard form:

$$C(t) = 4e^{0.06t-1} = \frac{4e^{0.06t}}{e} = \left(\frac{4}{e}\right)\left(e^{0.06}\right)^t$$

So $C(t)$ has yearly growth factor $e^{0.06}$. This implies that the yearly percent growth rate is $100\left(e^{0.06}-1\right)\%$; that is, the population of crows grows by $\boxed{100\left(e^{0.06}-1\right)\%}$ every year.

e. [5 points] When will there be the same number of lizards and crows on the island?

Solution:
Recall that the populations of lizards and crows (measured in thousands) t years after the island is discovered are given by the formulas $L(t) = 5(1-0.05)^t$ and $C(t) = 4e^{0.06t-1}$, respectively. We can use these formulas to solve the equation $C(t) = L(t)$ for t:

$$\begin{aligned} 5(1-0.05)^t &= 4e^{0.06t-1} \\ \ln\left(5 \cdot 0.95^t\right) &= \ln\left(4e^{0.06t-1}\right) \\ \ln(5) + t \cdot \ln(0.95) &= \ln(4) + 0.06t - 1 \\ t \cdot \ln(0.95) - 0.06t &= \ln(4) - \ln(5) - 1 \\ t(\ln(0.95) - 0.06) &= \ln(4) - \ln(5) - 1 \\ t &= \frac{\ln(4) - \ln(5) - 1}{\ln(0.95) - 0.06} \end{aligned}$$

Therefore, the populations of lizards and crows on the island will agree $\boxed{\dfrac{\ln(4) - \ln(5) - 1}{\ln(0.95) - 0.06}}$ years after the island is discovered.